MATH 2058 Mathematical Analysis I
2023-24 Term 1
Suggested Solution to Homework 3

3.4-10 Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n \coloneqq \sup\{x_k : k \geq n\}$ and $S \coloneqq \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.

Solution. First we note that (s_n) is a bounded decreasing sequence. Hence $\lim(s_n) = S$ by the Monotone Convergence Theorem.

By the definition of s_1, there exists $n_1 \geq 1$ such that $s_1 - \frac{1}{1} < x_{n_1} \leq s_1$.

Suppose $n_k \geq k$ is chosen. By the definition of s_{n_k+1}, there exists $n_{k+1} \geq n_k + 1 \geq k + 1$ such that $s_{n_k+1} - \frac{1}{k+1} < x_{n_{k+1}} \leq s_{n_k+1}$.

Continue in this way, we find a sequence (n_k) in $\mathbb{N}$ such that for all $k \in \mathbb{N}$, $n_{k+1} > n_k$, $n_k \geq k$ and

$$s_{n_k+1} - \frac{1}{k+1} < x_{n_{k+1}} \leq s_{n_k+1}.$$

Since $\lim\left(s_{n_k+1} - \frac{1}{k+1}\right) = \lim(s_{n_k+1}) = S$, it follows from the Squeeze Theorem that the subsequence (x_{n_k}) of (x_n) converges to S.

□

3.4-12 Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

Solution. As (x_n) is unbounded, we have for any $M > 0$, there is $n \in \mathbb{N}$ such that $|x_n| > M$.

Pick $n_1 \in \mathbb{N}$ such that $|x_{n_1}| > 1$.
Then pick $n_2 \in \mathbb{N}$ such that $|x_{n_2}| > \max\{2, |x_1|, |x_2|, \ldots, |x_{n_1}|\}$. So $|1/x_{n_2}| < 1/2$ and $n_2 > n_1$.

Suppose $n_1 < n_2 < \cdots < n_k$ are chosen so that $|1/x_{n_j}| < 1/j$ for $1 \leq j \leq k$.

Pick $n_{k+1} \in \mathbb{N}$ such that $|x_{n_{k+1}}| > \max\{k+1, |x_1|, |x_2|, \ldots, |x_{n_k}|\}$. So $|1/x_{n_{k+1}}| < 1/(k+1)$ and $n_{k+1} > n_k$.
Continue in this way, we obtain a subsequence (x_{n_k}) of (x_n) such that

$$|1/x_{n_k}| < 1/k \qquad \text{for all } k \in \mathbb{N}.$$

Now $\lim(1/x_{n_k}) = 0$ follows immediately from the Squeeze Theorem. □