MATH 2058 Mathematical Analysis I
2023-24 Term 1
Suggested Solution to Homework 2

3.1-5 Use the definition of the limit of a sequence to establish the following limits.

(d) $\lim\left(\dfrac{n^2-1}{2n^2+3}\right)=\dfrac{1}{2}$.

Solution. Note that

$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right|=\frac{5}{4n^2+6}<\frac{5}{4n^2}<\frac{2}{n^2}\qquad\text{for any } n\in\mathbb{N}.$$

So for any $\varepsilon>0$, we choose a natural number K such that $K>\sqrt{\frac{2}{\varepsilon}}$ by the Archimedean Property. Then for any $n\geq K$, we have

$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right|<\frac{2}{n^2}\leq\frac{2}{K^2}<\varepsilon.$$

Hence

$$\lim\left(\frac{n^2-1}{2n^2+3}\right)=\frac{1}{2}.$$

□

3.1-8 Prove that $\lim(x_n)=0$ if and only if $\lim(|x_n|)=0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).

Solution. First, we show a slightly stronger statement: $\lim(x_n)=\ell \implies \lim(|x_n|)=|\ell|$. For any $\varepsilon>0$, we can find a natural number K such that for any $n\geq K$, we have

$$|x_n-\ell|<\varepsilon$$

by the definition of $\lim(x_n)=\ell$. Hence, by the reverse triangle inequality, we have,

$$||x_n|-|\ell||\leq|x_n-\ell|<\varepsilon\qquad\text{for any } n\geq K.$$

So $\lim(|x_n|)=|\ell|$.

Next, we show the converse: $\lim(|x_n|)=0 \implies \lim(x_n)=0$. Again, for any $\varepsilon>0$, we can find a natural number K, such that for any $n\geq K$, we have $||x_n|-0|<\varepsilon$, which is just $|x_n|<\varepsilon$. Hence, for $n\geq K$, we have

$$|x_n-0|=|x_n|<\varepsilon.$$

So $\lim(x_n)=0$.

Consider $(x_n)=((-1)^n)$. Clearly (x_n) does not converge but $(|x_n|)=(|(-1)^n|)=(1)$, which is a constant sequence and converges. □

3.2-23 Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent.

Solution. By Exercise 2.2.18 in the textbook, if $a, b \in \mathbb{R}$, then

$$\max\{a,b\} = \frac{1}{2}(a+b+|a-b|) \quad \text{and} \quad \min\{a,b\} = \frac{1}{2}(a+b-|a-b|). \tag{1}$$

By Proposition 2.8 (of the lecture notes) and the proof in the previous question, $(|x_n - y_n|)$ is convergent. By (1) and Proposition 2.8 again, (u_n) and (v_n) are also convergent.

□

3.3-3 Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

Solution. First, it is easy to see from induction that $x_n \geq 2$ for all $n \in \mathbb{N}$.

Now for all $n \in \mathbb{N}$, since $x_n - 1 \geq 1$, we have

$$x_{n+1} - 1 = \sqrt{x_n - 1} \leq x_n - 1.$$

So $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, and (x_n) is a decreasing sequence.

Theorem 2.13 (Monotone Convergence Theorem) then implies that (x_n) is convergent. Suppose $A = \lim(x_n)$. Then $A \geq 2$ by Proposition 2.9 (of the lecture notes), and it satisfies

$$(A-1)^2 = A - 1.$$

So $A = 2$ or $A = 1$. The latter is impossible since $A \geq 2$. Therefore $\lim(x_n) = 2$. □