MATH 2058 Mathematical Analysis I
2023-24 Term 1
Suggested Solution to Homework 1

Refer to textbook: R.G. Bartle and I.D. R. Sherbert, Introduction to real analysis, 4th edition, Wiley

2.4-7 Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A+B \coloneqq \{a+b : a \in A, b \in B\}$. Prove that $\sup(A+B) = \sup A + \sup B$ and $\inf(A+B) = \inf A + \inf B$.

Solution. We will only prove $\inf(A+B) = \inf A + \inf B$ here as $\sup(A+B) = \sup A + \sup B$ can be proved similarly.

By the Axiom of Completeness, both $\inf A$ and $\inf B$ exist. It is clear that $\inf A + \inf B$ is a lower bound of $A+B$.

Let $\varepsilon > 0$. By Theorem 1.5*(ii')*, there are $a_0 \in A$ and $b_0 \in B$ such that $a_0 < \inf A + \frac{\varepsilon}{2}$ and $b_0 < \inf B + \frac{\varepsilon}{2}$. Hence $a_0 + b_0 < \inf A + \inf B + \varepsilon$.

By Theorem 1.5 again, we have $\inf(A+B) = \inf A + \inf B$. □

2.4-8 Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

and that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x)+g(x) : x \in X\}.$$

Give examples to show that each of these inequalities can be either equalities or strict inequalities.

Solution. We will only prove the first inequality. The second one can be proved similarly.

Denote $a \coloneqq \sup\{f(x) : x \in X\}$ and $b \coloneqq \sup\{g(x) : x \in X\}$, which exist because f and g have bounded ranges in $\mathbb{R}$. For any $x \in X$, we have $f(x) \leq a$ and $g(x) \leq b$, and hence $f(x)+g(x) \leq a+b$. So $a+b$ is an upper bound of the set $\{f(x)+g(x) : x \in X\}$,which implies that

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

For equality, we can simply take $X = \mathbb{R}$ and $f = g = 1$.

For strict inequality, take $X = [-1, 1]$ and $f(x) = -g(x) = x$ for any $x \in X$. Then $\sup\{f(x) + g(x) : x \in X\} = 0$ while $\sup\{f(x) : x \in X\} = \sup\{g(x) : x \in X\} = 1$. □

2.4-9 Let $X = Y \coloneqq \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) \coloneqq 2x + y$.

(a) For each $x \in X$, find $f(x) \coloneqq \sup\{h(x, y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.

(b) For each $y \in Y$, find $g(y) \coloneqq \inf\{h(x, y) : x \in X\}$; then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).

Solution. (a) Fix $x \in X$. By Exercise 2.4-6, we have

$$f(x) = \sup\{2x + y : y \in (0,1)\} = 2x + \sup\{y : y \in (0,1)\} = 2x + 1.$$

So, by Exercise 2.4-4 and 2.4-6, we have

$$\inf\{f(x) : x \in X\} = \inf\{2x + 1 : x \in (0,1)\} = 2\inf\{x : x \in (0,1)\} + 1 = 2 \cdot 0 + 1 = 1.$$

(b) Fix $y \in Y$. By Exercise 2.4-4 and 2..4-6, we have

$$g(y) = \inf\{2x + y : x \in (0,1)\} = 2\inf\{x : x \in (0,1)\} + y = 2 \cdot 0 + y = y.$$

So,

$$\sup\{g(y) : y \in Y\} = \sup\{y : y \in (0,1)\} = 1.$$

The results in (a) and (b) are the same.

$\square$