MATH2050C Selected Solution to Assignment 6

Section 3.5

(5) Using $\sqrt{n+1}-\sqrt{n}=1/(\sqrt{n+1}+\sqrt{n})$, it is easy to see that $x_n \to 0$ as $n \to \infty$. However, since x_n is unbounded, it cannot be a Cauchy sequence. (Every Cauchy sequence is convergent and hence is necessarily bounded.)

(13) It is clear that $2 \le x_n$ for all n. Therefore,

$$|x_{n+2}-x_{n+1}| = \frac{1}{x_{n+1}x_n}|x_{n+1}-x_n| \le \frac{1}{2} \times \frac{1}{2}|x_{n+1}-x_n| \ ,$$

hence $\{x_n\}$ is contractive with $\gamma = 1/4$. Consequently its limit exists and is the positive root of $x = 2 + 1/x$ which is $x = 1+\sqrt{2}$.

(14) Define $x_{n+1} = (x_n^3+1)/5$. By induction one can show that $0 < x_n < 2/5,\ n \ge 2$, if $x_1 \in (0,1)$. Then

$$|x_{n+2}-x_{n+1}| = \frac{1}{5}|x_{n+1}^3-x_n^3| = |x_{n+1}^2+x_{n+1}x_n+x_n^2||x_{n+1}-x_n| \le \frac{3}{5}\times\frac{4}{25}|x_{n+1}-x_n| = \frac{12}{125}|x_{n+1}-x_n| \ ,$$

so x_n is contractive. Its limit is a root of $x^3-5x+1=0$ in the range $(0,2/5)$.

Supplementary Problems

1. Define the Fibonacci sequence by $f_{n+2} = f_{n+1}+f_n, f_1 = f_2 = 1$. Show that

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Hint: $(1\pm\sqrt{5})/2$ are two roots of $x^2 = x+1$ and f_n should be given by their linear combination.

Solution Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$ be the two roots of $x^2 = x+1$. Then $a^{n+2} = a^{n+1}+a^n$ and $b^{n+2} = b^{n+1}+b^n$ so for any constants c_1, c_2, $u_n = c_1a^n + c_2b^n$ satisfies $u_{n+2} = u_{n+1}+u_n$. Now it suffices to choose c_1, c_2 such that $u_1 = u_2 = 1$. That is, to solve the linear system $ac_1+bc_2 = 1, a^2c_1+b^2c_2 = 1$. It is straightforward to get $c_1 = 1/\sqrt{5},\ c_2 = -1/\sqrt{5}$.