MATH2050C Selected Solution to Assignment 5

Section 3.4

(4a). The subsequence $b_n = a_{2n} = 1/(2n) \to 0$ as $n \to \infty$. On the other hand, the subsequence $c_n = a_{2n+1} = 2 + 1/(2n+1) \to 2$ as $n \to \infty$. Since these two subsequences converge to different limits, $\{a_n\}$ is divergent.

(b). The subsequence $b_k = a_{8k} = \sin 8k\pi/4 = 0$ while the subsequence $c_k = a_{8k+2} = \sin(8k + 2)\pi/4 = 1$. Thus the first subsequence tends to 0 and the second one to 1. We conclude that this sequence is divergent.

(7a). Observe $a_n = (1 + 1/n^2)^{n^2}$ is a subsequence of $c_n = (1 + 1/n)^n$. In fact, $a_n = c_{n^2}$. Since every subsequence converges to the same limit for a convergent sequence, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = e$.

(d). In a previous exercise we have shown that $a_n = (1+2/n)^n$ is convergent (actually when 2 is replaced by any positive a). Denote its limit by a. Then the subsequence $b_k = a_{2k} = (1+1/k)^{2k}$ should tend to the same a. But now it is clear that it converges to e^2, so $a = e^2$. Therefore, $\lim_{n\to\infty} a_n = e^2$.

(9). If $\{x_n\}$ does not converge to 0, for some $\varepsilon_0 > 0$, there are $n_j \to \infty$ such that $|x_{n_j} - 0| \geq \varepsilon_0$. Thus $\{x_{n_j}\}$ cannot converge to 0.

(11). Let $a_n = (-1)^n x_n$. By assumption it tends to some a. The subsequence $b_k = a_{2k} = x_{2k}$ tends to a, showing that $a \geq 0$. On the other hand, $c_k = a_{2k+1} = -x_{2k+1}$ also tends to a, showing that $a \leq 0$. (Recall it is assumed that all $x_n \geq 0$.) We conclude that $a = 0$. For every $\varepsilon > 0$, there is some n_ε such that $|x_n - 0| = |(-1)^n x_n - 0| < \varepsilon$ for all $n \geq n_\varepsilon$, hence $\{x_n\}$ converges to 0.

Supplementary Exercises

1. Can you find a sequence from $[0, 1]$ with the following property: For each $x \in [0, 1]$, there is subsequence of this sequence taking x as its limit? Suggestion: Consider the rational numbers.

   Solution Let $\{r_n\}$ be an enumeration of the set of all rational numbers in $[0, 1]$. This is possible as all rational numbers form a countable set. Let $x \in [0, 1]$. We claim that it is a limit point. For each $n \geq 1$, there are infinitely many rational numbers in $(x - 1/n, x + 1/n) \cap [0, 1]$. We can pick one by one from $\{r_n\}$ to form $\{r_{n_k}\}$ so that $n_k < n_{k+1}$, that is, $\{r_{n_k}\}$ is a subsequence. Now, given $\varepsilon > 0$, pick some n_1 such that $1/n_1 < \varepsilon$. It then follows that for all $n_k \geq n_1$, $|r_{n_k} - x| < 1/n_k \leq 1/n_1 < \varepsilon$. We conclude $r_{n_k} \to x$.

   Note. This exercise shows that the set of limit points of a single sequence could be very large.

2. Recall that for $a \geq 0$, $E(a) = \lim_{n\to\infty}(1 + a/n)^n$ is well-defined. Show that for a rational $a > 0$, $E(a) = e^a$.

Solution Let $a = p/q$. We have

$$\left(1 + \frac{p/q}{kp}\right)^{kp} = \left(\left(1 + \frac{1}{qk}\right)^{qk}\right)^{p/q} .$$

Since $x_n = (1 + 1/n)^n$ converges to e, so does the subsequence $y_k = x_{qk}$. Letting $k \to \infty$ and using the result proved in Supp. Problem 3 in Ex 3: $x_n \to x$ implies $x_n^{p/q} \to x^{p/q}$,

$$\begin{aligned} E(p/q) &= \lim_{k\to\infty} \left(1 + \frac{p/q}{kp}\right)^{kp} \\ &= \lim_{k\to\infty} \left(\left(1 + \frac{1}{qk}\right)^{qk}\right)^{p/q} \\ &= \left(\lim_{k\to\infty} \left(1 + \frac{1}{qk}\right)^{qk}\right)^{p/q} \\ &= e^{p/q} . \end{aligned}$$

Note After a meaning is assigned to e^a for irrational a's, one has $E(a) = e^a$ for all $a \in \mathbb{R}$. We will do this later.

3. Let $\{x_n\}$ be a positive sequence such that $a = \lim_{n\to\infty} x_{n+1}/x_n$ exists. Show that $\lim_{n\to\infty} x_n^{1/n}$ exists and is equal to a.

Solution Write

$$x_n = \frac{x_n}{x_{n-1}} \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_2}{x_1} x_1 .$$

For small $\varepsilon > 0$, let n_0 be $a - \varepsilon < x_n/x_{n-1} < a + \varepsilon$ for all $n \geq n_0$. Then

$$x_n \leq (a+\varepsilon)(a+\varepsilon) \cdots \frac{x_{n_0-1}}{x_{n_0-2}} \cdots \frac{x_2}{x_1} x_1 \leq (a+\varepsilon)^{n-n_0+1} K ,$$

where K depends on n_0. It follows that

$$x_n^{1/n} \leq (a+\varepsilon)^{(n-n_0+1)/n} K \leq (a+\varepsilon)(a+\varepsilon)^{(-n_0+1)/n} K^{1/n}.$$

We have a similar inequality from the other side:

$$(a-\varepsilon)(a-\varepsilon)^{(-n_0+1)/n} K^{1/n} \leq x_n^{1/n} .$$

It shows that $x_n^{1/n}$ is bounded. To show its limit exists, by Theorem 3.4.9 in Text or Theorem 5.2 in Ex 5, it suffices to show its convergent subsequences converge to the same limit. Let x_{n_j} be a convergent subsequence which converges to some $b \neq a$. Let $\varepsilon = |b-a|/2$. There is some j_0 such that $|x_{n_j}^{1/n_j} - b| < |b-a|/2$ for all $j \geq j_0$. Passing to infinity in the above inequalities for x_{n_j}, we get $(a-\varepsilon) \leq b \leq (a+\varepsilon)$, that is, $|b-a| \leq |b-a|/2$, contradiction holds.

Remark The above proof aims to illustrate the use of Theorem 3.4.9. A student suggests to me the following direct proof. Looking at the inequality,

$$(a-\varepsilon)(a-\varepsilon)^{(-n_0+1)/n} K^{1/n} \leq x_n^{1/n} \leq (a+\varepsilon)(a+\varepsilon)^{(-n_0+1)/n} K^{1/n} ,$$

and noting $a^{1/n} \to 1$, for the same ε, one can find another $n_1, n_1 \geq n_0$, such that

$$(a \pm \varepsilon)^{(-n_0+1)/n} K^{1/n} < 1 + \varepsilon \ ,$$

for all $n \geq n_1$. It follows that

$$(1+\varepsilon)(a-\varepsilon) < x_n^{1/n} \leq (1+\varepsilon)(a+\varepsilon) \ ,$$

which implies

$$-C\varepsilon \leq x_n^{1/n} - a \leq C\varepsilon, \quad n \geq n_1 \ ,$$

for some constant C.

4. Show that $\lim_{n\to\infty} \dfrac{n}{(n!)^{1/n}} = e$.

   Solution Let $x_n = n^n/n!$ so that $\dfrac{n}{(n!)^{1/n}} = x_n^{1/n}$. Now, $x_{n+1}/x_n = (1+1/n)^n \to e$ and the desired conclusion follows from the result in Problem 3.

   Remark A formula that relates $n!$ to n^n is given by the Stirling's formula: $n! \sim \sqrt{2\pi n}(n/e)^n$.

5. The concept of a sequence extends naturally to points in $\mathbb{R}^N$. Taking $N = 2$ as a typical case, a sequence of ordered pairs, $\{\mathbf{a}_n\}, \mathbf{a}_n = (x_n, y_n)$, is said to be convergent to $\mathbf{a}$ if, for each $\varepsilon > 0$, there is some n_0 such that

$$|\mathbf{a}_n - \mathbf{a}| < \varepsilon \ , \quad \forall n \geq n_0 \ .$$

   Here $|\mathbf{a}| = \sqrt{x^2 + y^2}$ for $\mathbf{a} = (x, y)$. Show that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

   Solution It follows from the elementary inequalities

$$|x_1 - y_1|, |x_2 - y_2| \leq |\mathbf{a} - \mathbf{b}| \leq |x_1 - y_1| + |x_2 - y_2| \ ,$$

   which show that $\mathbf{a}_n \to \mathbf{a}$ if and only if $x_n \to x$ and $y_n \to y$.

6. Bolzano-Weierstrass Theorem in $\mathbb{R}^N$ reads as, every bounded sequence in $\mathbb{R}^N$ has a convergent subsequence. Prove it. A sequence is bounded if $|\mathbf{a}_n| \leq M, \ \forall n$, for some number M.

   Solution Take $N = 2$ for simplicity. $\{\mathbf{a}_n\}$ is bounded implies $\{x_n\}$ and $\{y_n\}$ are bounded by the previous exercise. Pick a convergent subsequence $\{x_{n_k}\}$ from $\{x_n\}$. As $\{y_{n_k}\}$ is a bounded sequence, pick a convergent sequence $\{y_{n_{k_j}}\}$ from $\{y_{n_k}\}$. Then $(x_{n_{k_j}}, y_{n_{k_j}})$ is a convergent subsequence for $\mathbf{a}_n = (x_n, y_n)$.