MATH2050C Selected Solutions to Assignment 4

Section 3.1

(5d) We have

$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right|=\frac{5}{2(2n^2+3)}<\frac{5}{4n^2}\ .$$

Therefore, for any number n_ε satisfying $\geq \sqrt{[5/4\varepsilon]+1}$, we have

$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right|<\varepsilon\ ,\quad \forall n\geq n_\varepsilon\ .$$

So

$$\lim_{n\to\infty}\frac{n^2-1}{2n^2+3}=\frac{1}{2}\ .$$

Here $[a]$ is the integer part of a. For instance, $[1.2]=1, [5]=5, [-3.4]=-3$.

(6c) Using

$$0<\frac{\sqrt{n}}{n+1}<\frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}\ ,$$

we see that for all $n\geq[1/\varepsilon^2]+1$,

$$\left|\frac{\sqrt{n}}{n+1}-0\right|<\frac{1}{\sqrt{n}}<\varepsilon\ ,\quad \forall n\geq n_\varepsilon\ ,$$

where n_ε can be chosen to be any natural number $\geq[1/\varepsilon^2]+1$. So

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+1}=0\ .$$

(17) Use

$$\frac{2^n}{n!}=\frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{4}\cdots\frac{2}{n}<\frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{3}\cdots\frac{2}{3}=2\left(\frac{2}{3}\right)^{n-2}\ ,\quad \forall n\geq 4.$$

It suffices to choose n_ε such that

$$2\left(\frac{2}{3}\right)^{n-2}<\varepsilon,$$

that is,

$$n_\varepsilon>2+\frac{\log(\varepsilon/2)}{\log(2/3)}\ .$$

Note. In general, one can show that $\lim_{n\to\infty}a^n/n!=0$ for every $a>0$.

(18) It suffices to consider $\varepsilon=x/2$. Then there is some K such that $|x_n-x|<x/2,\quad \forall n\geq K$. By writing it as $-x/2<x_n-x<x/2$, we get $x/2<x_n<3x/2$.

Section 3.2 (1d) We write

$$\frac{2n^2+3}{n^2+1}=\frac{2+3/n^2}{1+1/n^2}.$$

Then

$$\begin{aligned}\lim_{n\to\infty}\frac{2n^2+3}{n^2+1} &= \lim_{n\to\infty}\frac{2+3/n^2}{1+1/n^2}\\ &= \frac{\lim_{n\to\infty}(2+3/n^2)}{\lim_{n\to\infty}(1+1/n^2)} \quad \text{(by Limit Theorem)}\\ &= \frac{2}{1}=2.\end{aligned}$$

(5) Both sequences are not bounded, so they cannot be convergent.

(11) (a) Write

$$(3n^{1/2})^{1/2n} = (3^{1/2})^{1/n}n^{1/4n} = (3^{1/2})^{1/n}(4n)^{1/4n}(4^{-1/4})^{1/n} = (3^{1/2}4^{-1/4})^{1/n}(4n)^{1/4n} .$$

Use the known facts $\lim_{n\to\infty} a^{1/n} = 1$ $(a>0)$ and $\lim_{n\to\infty} n^{1/n} = 1$, we have

$$\lim_{n\to\infty}(3n^{1/2})^{1/2n} = \lim_{n\to\infty} a^{1/n}(4n)^{1/4n} = \lim_{n\to\infty} a^{1/n} \lim_{n\to\infty}(4n)^{1/4n} = 1\times 1 = 1,$$

where $a = 3^{1/2}4^{-1/4}$ by Limit Theorem.

Note. Here we have used the trivial fact: $\lim_{n\to\infty} n^{1/n} = 1$ implies $\lim_{n\to\infty}(4n)^{1/4n} = 1$.

(b) Let $x_n = (n+1)^{1/\log(n+1)}$. Then $\log x_n == \dfrac{1}{\log(1+n)}\log(1+n) = 1$. So this is a constant sequence $\{e, e, e, \cdots\}$ and $\lim_{n\to\infty} x_n = e$.

(12) We have

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{b^{n+1}(1+(a/b)^{n+1})}{b^n(1+(a/b)^n)} = b\frac{1+(a/b)^{n+1}}{1+(a/b)^n} .$$

Therefore, by Limit Theorem, and $0 < a/b < 1$,

$$\lim_{n\to\infty}\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \lim_{n\to\infty} b\frac{1+(a/b)^{n+1}}{1+(a/b)^n} = b\frac{\lim_{n\to\infty}(1+(a/b)^{n+1})}{\lim_{n\to\infty}(1+(a/b)^n)} = b .$$

(as $\lim_{n\to\infty}(a/b)^n = 0$ for $a/b \in (0,1)$.)

Note. We have used the fact $\lim_{n\to\infty} \alpha^n = 0$ for $\alpha \in (0,1)$. The fact was proved in class and in the text book. You may simply quote it.

Supplementary Exercise

(1) Find the limit of $\{x_n\}, x_n = \frac{7n^2+3}{n^2-n-5}$. Determine n_0 explicitly for given $\varepsilon > 0$. Recall definition: $\{x_n\}$ converges to x if for each $\varepsilon > 0$, there is some n_0 such that $|x_n - x| < \varepsilon$ for all $n \geq n_0$.

Solution We guess the limit is 7.

$$\begin{aligned}
\left|\frac{7n^2+3}{n^2-n-5}-7\right| &= \left|\frac{7n+38}{n^2-n-5}\right| \\
&= \left|\frac{7n+38}{1/2n^2+1/2n^2-n-5}\right| \\
&\le \left|\frac{7n+38}{1/2n^2}\right| , \quad n \ge 5, \\
&\le \frac{14}{n}+\frac{76}{n^2} .
\end{aligned}$$

As both $14/n, 76/n^2 \to 0$ as $n \to \infty$, $14/n + 76/n^2 \to 0$ also holds. For $\varepsilon > 0$, there is some n_0 such that $14/n + 76/n^2 < \varepsilon$ for all $n \ge n_0$. Taking n_0 further satisfying $n_0 \ge 5$, we conclude $|(7n^2+3)/(n^2-n-5) - 7| < \varepsilon$ for $n \ge n_0$.

(2) Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n, a_n \neq 0$, and $q(x) = b_0 + b_1 x + \cdots + b_m x^m, b_m \neq 0$, be two polynomials. Consider the sequence $x_k = p(k)/q(k), k \ge 1$, (when k is large, $q(k)$ does not vanish, so you may assume that q is always non-zero). Prove that (a) When $n = m$, $\lim_{k\to\infty} x_k = a_n/b_m$;

(b) When $n > m$, $\{x_k\}$ does not converge ; and

(c) When $n < m$, $\lim_{k\to\infty} x_k = 0$.

(a) Write

$$\frac{p(k)}{q(k)} = \frac{k^n(a_0/k^n + a_1/k^{n-1} + \cdots + a_n)}{k^m(b_0/k^m + b_1/k^{m-1} + \cdots + b_m)} = \frac{a_0/k^n + a_1/k^{n-1} + \cdots + a_n}{b_0/k^n + b_1/k^{n-1} + \cdots + b_n},$$

when $m = n$. By Limit Theorem,

$$\lim_{k\to\infty} \frac{p(k)}{q(k)} = \frac{\lim_{k\to\infty}(a_0/k^n + a_1/k^{n-1} + \cdots + a_n)}{\lim_{k\to\infty}(b_0/k^n + b_1/k^{n-1} + \cdots + b_n)} = \frac{a_n}{b_n} .$$

(b) WLOG let $a_n, b_m > 0$. Using the fact that $\lim_{n\to\infty}(a_0/k^n + a_1/k^{n-1} + \cdots + a_{n-1}/k) = 0$, for $\varepsilon > 0$, there is some k_0 such that

$$|a_0/k^n + a_1/k^{n-1} + \cdots + a_{n-1}/k - 0| < \varepsilon, \quad \forall n \ge n_0.$$

Choose $\varepsilon = a_n/2$, we have

$$|a_0/k^n + a_1/k^{n-1} + \cdots + a_{n-1}/k - 0| < \frac{a_n}{2}, \quad \forall k \ge k_0.$$

It follows that $a_0/k^n + a_1/k^{n-1} + \cdots + a_{n-1}/k + a_n > a_n/2$. Similarly we can find k_1 such that $b_0/k^m + b_1/k^{m-1} + \cdots b_{m-1}/k + b_m < 2b_m$ for all $k \ge k_1$. Thus,

$$\frac{p(k)}{q(k)} = \frac{k^n(a_0/k^n + a_1/k^{n-1} + \cdots + a_n)}{k^m(b_0/k^m + b_1/k^{m-1} + \cdots + b_m)} > \frac{k^n a_n/2}{k^m 2b_m} = \frac{a_n}{4b_m} k^{n-m}$$

for all $k \ge \max\{k_0, k_1\}$. Now, given $M > 0$, it is clear there is some K such that

$$\frac{p(k)}{q(k)} \ge \frac{a_n}{4b_m} k^{n-m} > M$$

for all $k \geq K$. Indeed, it suffices to choose to be any natural number satisfying

$$K \geq k_0, k_1, \left(\frac{4b_m M}{a_n}\right)^{1/(n-m)} .$$

We conclude that $\{x_k\}$ is not convergent, in fact,

$$\lim_{k\to\infty} x_k = \infty.$$

(When $a_n b_m < 0$, it is $-\infty$ instead of ∞.)
(c) Similar to (a) we have

$$\frac{p(k)}{q(k)} = \frac{k^n(a_0/k^n + a_1/k^{n-1} + \cdots + a_n)}{k^m(b_0/k^m + b_1/k^{m-1} + \cdots + b_m)} = \frac{1}{k^{m-n}} \frac{a_0/k^n + a_1/k^{n-1} + \cdots + a_n}{b_0/k^m + b_1/k^{m-1} + \cdots + b_m}.$$

By Product Rule,

$$\lim_{k\to\infty} \frac{p(k)}{q(k)} = \lim_{k\to\infty} \frac{1}{k^{m-n}} \lim_{k\to\infty} \frac{a_0/k^n + a_1/k^{n-1} + \cdots + a_n}{b_0/k^m + b_1/k^{m-1} + \cdots + b_m} = 0 \times \frac{a_n}{b_m} = 0 .$$

(3) Suppose that $x_n \to x, x_n \geq 0$. Show that $x_n^{p/q} \to x^{p/q}$ for $p, q \in \mathbb{N}$.

Solution Assume $x > 0$. We show $x_n^{1/q} \to x^{1/q}$ if $x_n \to x$. Indeed, from

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}),$$

we get

$$x_n^{1/q} - x^{1/q} = \frac{x_n - x}{x_n^{1-1/q} + x_n^{1-2/q}x^{1/q} + \cdots + x^{1-1/q}} .$$

For $\varepsilon = x/2$, there is some n_0 such that $|x_n - x| < x/2$, which implies $x_n \geq x/2$ for $n \geq n_0$. Since there are q-many terms in the denominator, $x_n^{1-1/q} + x_n^{1-2/q}x^{1/q} + \cdots + x^{1-1/q} \geq q \times (x/2)^{1-1/q} \equiv \alpha$. It follows that $|x_n^{1/q} - x^{1/q}| \leq \dfrac{1}{\alpha}|x_n - x|$ for $n \geq n_0$. For $\alpha\varepsilon > 0$, there is n_1 such that $|x_n - x| < \alpha\varepsilon$ for all $n \geq n_1$. Putting together, for $n \geq n_2 \equiv \max\{n_0, n_1\}$,

$$|x_n^{1/q} - x^{1/q}| \leq \frac{1}{\alpha}|x_n - x| < \frac{1}{\alpha} \times \alpha\varepsilon = \varepsilon .$$

By the Product Rule, as $x_n^{1/q} \to x^{1/q}$, $x_n^{p/q} \to x^{p/q}$ too.
When $x = 0$, for $\varepsilon^{q/p} > 0$, there is some n_0 such that $0 \leq x_n < \varepsilon^{q/p}$ for all $n \geq n_0$. It follows that $0 \leq x_n^{p/q} < \varepsilon$ for $n \geq n_0$, done.