February 26, 2024

MATH2050C Selected Solution to Midterm Examination

Answer all five questions. Justify your answer.

1. (a) (5 marks) Define the supremum for a nonempty set S in $\mathbb{R}$.
   (b) (5 marks) State the Completeness Property of $\mathbb{R}$.
   (c) (10 marks) Use (b) to prove the Archimedean Property: For any positive $x \in \mathbb{R}$ there is a natural number n satisfying $0 < x < n$.

2. (10 marks) Suppose that the sequence $\{a_n\}, n \geq 1$, converges to a. Show that the sequence $\{b_n\}$ given by

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \ ,$$

   also converges to a.

3. (10 marks) Determine the limit of $(1 + 1/n^2)^{5n^2}$ as $n \to \infty$. You may use the fact $e = \lim_{n\to\infty}(1 + 1/n)^n$.

4. (a) (5 marks) Define a Cauchy sequence.
   (b) (15 marks) Prove that every Cauchy sequence converges. Hint: Use Bolzano-Weierstrass Theorem.

5. Show that the following sequences are convergent as n goes to ∞ and find their limits (except (c)).

   (a) (10 marks)

$$x_n = \frac{2n^2 - 2n + 6}{5n^2 + n - 7} \ .$$

   Solution Since $2 - 2/n + 6/n^2 \to 2$ and $5 + 1/n - 7/n^2 \to 5$ as $n \to \infty$, by Limit Theorem we have

$$\lim_{n\to\infty} \frac{2n^2 - 2n + 6}{5n^2 + n - 7} = \lim_{n\to\infty} \frac{2 - 2/n + 6/n^2}{5 + 1/n - 7/n^2} = \frac{2}{5} \ .$$

   (b) (10 marks)

$$y_n = 2^{1/n} \ .$$

   Solution Write $2^{1/n} = 1 + d_n, d_n > 0$. By Bernoulli's inequality, $2 = (1 + d_n)^n \geq 1 + nd_n$ which implies $d_n \leq 1/n$. Therefore, $2^{1/n} - 1 = d_n \to 0$, too.

   Another proof. As $\frac{2^{1/(n+1)}}{2^{1/n}} = \frac{1}{2^{1/n(n+1)}} < 1$ which implies that $\{y_n\}$ is decreasing. As $y_n \geq 1$, Monotone Convergence Theorem asserts that the limit of y_n exists. Denoting it by a, from $2^{1/n}, 2^{1/2n} \to a$ we deduce $a = a^{1/2}$, so $a = 1$.

   Note Cannot not take log to get $\ln y_n = \ln 2/n \to 0$. We have not yet defined the log function.

(c) (10 marks)

$$z_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \ .$$

Solution First it is clear that z_n is increasing. By Monotone Convergence Theorem it suffices to show its boundedness from above. In fact, using $1/n^2 < 1/n(n-1) = 1/(n-1) - 1/n$ we have

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{n\times(n-1)} = 1 + 1 - 1/n < 2 \ .$$

(d) (10 marks) $\{a_n\}$ defined by

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{7}{a_n}\right), \ a_1 = 1 \ .$$

Solution First of all, by AM-GM Inequality, we have $a_n \geq \sqrt{a_n \times 7/a_n} = \sqrt{7}$. Next,

$$a_{n+1} - a_n = \frac{1}{2}\left(\frac{7 - a_n^2}{a_n}\right) \leq 0 \ ,$$

so $\{a_n\}$ is decreasing. By Monotone Convergence Theorem, $a_n \to a$ for some a. Passing to limit in the relation $a_{n+1} = (a_n + 7/a_n)/2$ to get $a = (a + 7/a)/2$ so $a = \sqrt{7}$.