

MATH2050C Assignment 9

Deadline: March 26 , 2024.

Hand in: 4.3. no. 5ab, 11; Supp. Ex. 2, 5.

Section 4.3 no. 3, 4, 5abedh, 8, 11.

Supplementary Problems

Justify your answers in the following problems.

1. Evaluate

$$\lim_{x \rightarrow 2^-} \frac{x^2 - 4}{x - 2}.$$

2. Evaluate

$$\lim_{x \rightarrow -3} \frac{x^2 - 2x - 15}{x + 3}.$$

3. Evaluate

$$\lim_{x \rightarrow \infty} \frac{\cos 1/x}{x}.$$

4. Find $\lim_{x \rightarrow 0^+} \frac{5x - \sqrt{x}}{\sqrt{x} - x^3}$ for $c = 0^+$ and ∞ .

5. Find $\lim_{x \rightarrow c} x^3 e^{1/x}$ where $c = 0^+, 0^-, \infty$, and $-\infty$.

See next page

Further Comments on Limits of Functions

First, we have studied limits of functions. For polynomials and rational functions, their limits are well understood. Indeed, let $r(x) = p(x)/q(x)$ be a rational function. We knew (1) it is well defined on the set $E = \{x \in \mathbb{R} : q(x) \neq 0\}$, (since a polynomial has at most finitely many roots, E is the union of finitely many open intervals.) (2) $\lim_{x \rightarrow c} r(x) = r(c)$ whenever c satisfies $q(c) \neq 0$. (Before the evaluation it is better to make sure that p and q are reduced, that is, they do not have common factor.)

In order to have more examples to work on, we need to introduce more functions. In this chapter the following functions are studied:

- The **square root** $f_1(x) = \sqrt{x}$. It is defined on $[0, \infty)$ and $\lim_{x \rightarrow c} \sqrt{x} = \sqrt{c}$ for all $c \geq 0$.
- The **(rational) power** $f_2(x) = x^{m/n}$. Generalizing the square root, it is known from the last chapter that for each $x \geq 0$, there is a unique $y \geq 0$ satisfying $y^n = x$. We write $y = x^{1/n}$ the n -th root of x . Then $x^{m/n} = (x^{1/n})^m$, $x \in [0, \infty)$, is well-defined for all $n, m \in \mathbb{N}$. We also define $x^{-m/n} = 1/x^{m/n}$. The square root function is a special rational power. We have the following general result: Let f be a non-negative function on A and c is a cluster point of A . Then

$$\lim_{x \rightarrow c} f(x)^{m/n} = f(c)^{m/n}.$$

- The **absolute value function** $f_3(x) = |f(x)|$. It is defined on $(-\infty, \infty)$ and

$$\lim_{x \rightarrow c} |f(x)| = |f(c)|,$$

for all $c \in \mathbb{R}$.

- The **exponential function** $f_4(x) = e^x$, $x \in \mathbb{R}$. The exponential function is defined to be the limit $e^x = \lim_{n \rightarrow \infty} (1 + x/n)^n$ which was proved to exist in a previous section. It is equal to

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

We have

$$\lim_{x \rightarrow c} e^x = e^c,$$

for all c . We provide a proof of this limit. First,

$$\left| e^x - \sum_{k=0}^n \frac{x^k}{k!} \right| = \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right| = \frac{x^{n+1}}{(n+1)!} \left| 1 + \frac{x}{n+2} + \frac{x^2}{(n+3)(n+2)} + \cdots \right|.$$

Therefore, for $x, |x| \leq M = |c| + 1$, we fix m such that $M/(m+2) \leq 1/2$,

$$\left| e^x - \sum_{k=0}^m \frac{x^k}{k!} \right| \leq \frac{M^{m+1}}{(m+1)!} \left(1 + \frac{M}{m+2} + \frac{M^2}{(m+2)^2} + \frac{M^3}{(m+2)^3} + \cdots \right).$$

It is easy to see that $M^{m+1}/(m+1)! \rightarrow 0$ as $m \rightarrow \infty$. For $\varepsilon > 0$, we can further assume m so large that $2M^{m+1}/(m+1)! < \varepsilon/3$. On the other hand, let $p(x) = \sum_{k=0}^m x^k/k!$. As $\lim_{x \rightarrow c} p(x) = p(c)$, for $\varepsilon > 0$, there is some $\delta < 1$ such that $|p(x) - p(c)| < \varepsilon/3$ for $0 < |x - c| < \delta$. Putting things together, we have

$$\begin{aligned} |e^x - e^c| &= |e^x - p(x) + p(x) - p(c) + p(c) - e^c| \\ &\leq |e^x - p(x)| + |p(x) - p(c)| + |p(c) - e^c| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

- The **sine function** $f_5(x) = \sin x, x \in \mathbb{R}$. Here the sine function is given by the formula

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots .$$

We derive from this formula the inequality $x - x^3/6 \leq \sin x \leq x$ for $0 \leq x \leq 1$ and use it to show

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 .$$

We also have

$$\lim_{x \rightarrow c} \sin x = \sin c ,$$

which can be proved by a similar method as in the case of the exponential function.

- The **cosine function** $f_6(x) = \cos x, x \in \mathbb{R}$. Here the cosine function is given by the formula

$$\sin x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

Using the basic relation $\sin^2 x + \cos^2 x = 1$ the formula $\sin x/x \rightarrow 1$ turns into

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} = -1/2.$$

We also have

$$\lim_{x \rightarrow c} \cos x = \cos c .$$

Second, variations on the notion of limits of functions including divergence at infinity and limits at infinity. Let f be function defined on $(a, b]$. It is said to **tend to ∞ (resp. $-\infty$) at a (from its right)** if for each $M > 0$, there is some $\delta > 0$ such that $f(x) > M$ (resp. $f(x) < -M$) for all $x \in (a, a + \delta)$. The notation is $\lim_{x \rightarrow a^+} f(x) = \infty$ (resp. $\lim_{x \rightarrow a^+} f(x) = -\infty$). Similarly, one can define $\lim_{x \rightarrow b^-} f(x) = \pm\infty$ (limit from the left). For f defined on (a, ∞) (resp. $(-\infty, b)$) we can define $\lim_{x \rightarrow \infty} f(x) = L$ if for each $\varepsilon > 0$ there is some $K > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > K$. Similarly, we can define $\lim_{x \rightarrow -\infty} f(x) = L$, $\lim_{x \rightarrow \infty} f(x) = \pm\infty$, $\lim_{x \rightarrow -\infty} f(x) = \pm\infty$, etc. For these variations of limits of functions, the corresponding Sequential Criterion, Limit Theorems, and Squeeze Theorem are for you to explore, or simply look up the text book.