MATH2050C Assignment 5

Deadline: Feb 20, 2024.

Hand in: 3.4 no 7; 3.5 no 3, 5, 9; Supp Ex no. 1.

Section 3.4 no. 4, 6, 8, 9, 11.

Supplementary Problems

1. Can you find a sequence from $[0,1]$ with the following property: For each $x \in [0,1]$, there is subsequence of this sequence taking x as its limit? Suggestion: Consider the rational numbers.

2. Recall that for $a \geq 0$, $E(a) = \lim_{n\to\infty}(1 + a/n)^n$ is well-defined. Show that for a rational $a > 0$, $E(a) = e^a$.

3. Let $\{x_n\}$ be a positive sequence such that $a = \lim_{n\to\infty} x_{n+1}/x_n$ exists. Show that $\lim_{n\to\infty} x_n^{1/n}$ exists and is equal to a.

4. Show that $\lim_{n\to\infty} \dfrac{n}{(n!)^{1/n}} = e$.

5. The concept of a sequence extends naturally to points in $\mathbb{R}^N$. Taking $N = 2$ as a typical case, a sequence of ordered pairs, $\{\mathbf{a}_n\}, \mathbf{a}_n = (x_n, y_n)$, is said to be convergent to $\mathbf{a}$ if, for each $\varepsilon > 0$, there is some n_0 such that

   $$|\mathbf{a}_n - \mathbf{a}| < \varepsilon \ , \quad \forall n \geq n_0 \ .$$

   Here $|\mathbf{a}| = \sqrt{x^2 + y^2}$ for $\mathbf{a} = (x, y)$. Show that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

6. Bolzano-Weierstrass Theorem in $\mathbb{R}^N$ reads as, every bounded sequence in $\mathbb{R}^N$ has a convergent subsequence. Prove it. A sequence is bounded if $|\mathbf{a}_n| \leq M$, $\forall n$, for some number M.

Bolzano-Weierstrass Theorem

Theorem 5.1 (Nested Interval Theorem). Let $I_j = [a_j, b_j], j \le 1$, be a sequence of closed intervals satisfying $I_{j+1} \subset I_j$. Then $\bigcap_j I_j = [a, b]$ where $a = \sup_j a_j$ and $b = \inf_j b_j$. In particular, the intersection of all I_j's are nonempty.

We refer to the Text for a proof, which is based on Monotone Convergence Theorem.

Theorem 5.2(Bolzano-Weierstrass Theorem). Every bounded sequence has a convergent subsequence.

Our proof is slightly different from the second proof in Text.

Proof. Let $\{x_n\}$ be a bounded sequence. Assume that it has infinitely many distinct points. (If not, the sequence is a finite set $\{a_1, a_2, \cdots, a_M\}$ one a_j's must repeatedly appear infinitely many times. You can choose this point to form a constant subsequence.) Fix a closed, bounded interval I_0 containing the sequence. We divide I_0 equally into two closed subintervals. Since the sequence has infinitely x_n's, one of these subintervals must contain infinitely many of them. Pick and call it I_1. Next, we divide I_1 equally into two closed subintervals and apply the same principle to pick I_2. Repeating this process, we end up with closed intervals $I_k, k \ge 1$, with the properties: For $k \ge 1$, (a) $I_{k+1} \subset I_k$, (b) the length of I_{k+1} is half that of I_k, and (c) there are infinitely points from $\{x_n\}$ sitting inside each I_k. Applying Nested Interval Theorem, $\cap_{k=1}^{\infty} I_k = \{x\}$. Now, we pick one $\{x_{n_k}\}$ from each I_k to form a subsequence. This is possible because there are infinitely many x_n's in each I_k. Clearly, $\{x_{n_k}\}$ converges to x.

A point a is called a **limit point** of the sequence $\{x_n\}$ if it is the limit of some subsequence of $\{x_n\}$. A bounded sequence has at least one limit point according to Bolzano-Weierstrass Theorem. A properly divergent sequence does not have any limit point. This following theorem is the same as Theorem 3.4.9 in Text.

Theorem 5.2. A bounded sequence is convergent if all its convergent subsequences have the same limit.

Proof. Assume that there is only one limit point x. Suppose on the contrary that the sequence does not converge to x. We can find some $\varepsilon_0 > 0$ and $n_k \to \infty$ such that $|x_{n_k} - x| \ge \varepsilon_0$. Since $\{x_{n_k}\}$ is bounded, it contains a subsequence $\{x_{n_{k_j}}\}$ which converges to some y satisfying $|y - x| = \lim_{j\to\infty} |x_{n_{k_j}} - x| \ge \varepsilon_0$. Since any subsequence of a subsequence is a subsequence of the original sequence, $\{x_{n_{k_j}}\}$ is again a subsequence of $\{x_n\}$. Thus y is a limit point different from x, contradiction holds.

Let $\{x_n\}$ be a bounded sequence. For each $n \ge 1$, the number

$$z_n = \sup_{k \ge n} x_k = \sup\{x_n, x_{n+1}, x_{n+2}, \cdots\} ,$$

is a number. It is clear that $\{z_n\}$ is decreasing and bounded from below. By Monotone Convergence Theorem, its limit exists. We call it the **limit superior** of the sequence of $\{x_n\}$. In notation,

$$\overline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \inf\{z_n\} = \inf_n \sup_{k \ge n} x_k .$$

Similarly, the number

$$w_n = \inf_{k \ge n} x_k = \inf\{x_n, x_{n+1}, x_{n+2}, \cdots\} ,$$

is a number. It is clear that $\{w_n\}$ is increasing and bounded from above. By Monotone Convergence Theorem, its limit exists. We call it the **limit inferior** of the sequence of $\{x_n\}$. In notation,

$$\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} w_n = \sup\{w_n\} = \sup_n \inf_{k\geq n} x_k \ .$$

Theorem 6.2. For a bounded sequence, its supremum is its largest limit point and its infimum the smallest limit point.

The following proof may be skipped in a first reading.

Proof *. Let b be the supremum of all limit points of $\{x_n\}$ and $a = \limsup_n x_n$. First, we claim that a is itself a limit point. Hence $a \leq b$. To do this we need to produce a subsequence convergence to a. For $\varepsilon = 1$, there is some n_0 such that $|z_n - a| < 1$ for all $n \geq n_0$. In particular, $|z_{n_0} - a| < 1$. Since $z_n = \sup\{x_n, x_{n+1}, x_{n+2}, \cdots, \}$, for the same $\varepsilon = 1$, there is some $m_0 \geq n_0$ such that $|x_{m_0} - z_{n_0}| < 1$. Next, by the same reasoning, for $\varepsilon = 1/2$, there is some $n_1 > n_0$ such that $|z_{n_1} - a| < 1/2$ and $m_1 \geq n_1$ such that $|z_{n_1} - x_{m_1}| < 1/2$. Continuing this, we obtain z_{n_k} and x_{m_k} where n_k and m_k are strictly increasing which satisfy $|z_{n_k} - a|, |z_{n_k} - x_{m_k}| < 1/k$. Therefore,

$$|x_{m_k} - a| \leq |x_{m_k} - z_{n_k}| + |z_{n_k} - a| < \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \ .$$

Letting $k \to \infty$, by Squeeze Theorem we conclude $\lim_{k\to\infty} x_{m_k} = a$, done.

On the other hand, to show $b \leq a$ it suffices to show $c \leq a$ for any limit point c. Let $c = \lim_{n_k\to\infty} x_{n_k}$ be such a limit point. For $\varepsilon > 0$, there is some n_{k_0} such that $c - \varepsilon < x_{n_k}$ for all $n_k \geq n_{k_0}$. As $x_k \leq z_k$ for all k, we have $c - \varepsilon \leq x_{n_k} \leq z_{n_k}$. Letting $k \to \infty$, $a = \lim_{n_k\to\infty} z_{n_k} \geq c - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $a \geq c$. Taking sup over c, we get $a \geq b$.

Now it is easy to show

Theorem 6.3. Let $\{x_n\}$ be a bounded sequence. Then

1. $\underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n$,
2. $\{x_n\}$ is convergent iff $\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n$. When this holds, $\lim_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$.