MATH2050C Assignment 2

Deadline: Jan 23, 2024.

Hand in: Section 2.4 no. 2, 8, 9, 14, 17.

Section 2.4 no. 1, 2, 7, 8, 9, 10, 14, 15, 17.

Supplementary Problems

1. Show for each positive number a and $n \geq 2$, there is a unique positive number b satisfying $b^n = a$. Suggestion: Use Binomial Theorem in Ex 1.

2. A real number is called an algebraic number if it is a root of some equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with integral coefficients. Show that the set of all algebraic numbers is a countable set containing all rational numbers and numbers of the form $a^{1/k}, a > 0, k \geq 1$.

See next page for a note on real numbers.

The Real Number System: Further Properties

From the Order Completeness Axiom (Completeness Property) we deduce

Archimedean Principle I. For any positive real number a, there is a natural number n such that $0 < a < n$.

Archimedean Principle II. For any positive real number ε, there is a natural number n such that $0 < 1/n < \varepsilon$.

Corollary 1 For any $a > 0$, there is an $m \in \mathbb{N}$ such that $m - 1 \le a < m$.

Corollary 2 For any real numbers $a, b,\ 0 < a < b$, there is a rational number lying strictly between a and b.

Immediately it implies there are infinitely many rational numbers strictly lying between a and b. Applying this to the numbers $a/\sqrt{2}$ and $b/\sqrt{2}$ it shows the same results hold when "rational" is replaced by "irrational".

Proof. It suffices to show there is some $m/n \in (a, b)$. In fact, since $b - a > 0$, by Archimedean Principle, there is some $n \in \mathbb{N}$ such that $b - a > 1/n$, that is, $n(b - a) > 1$, or $nb > na + 1$. On the other hand, Corollary 1 ensures that there is some m such that $m - 1 \le na < m$. It implies $a < m/n$. As $nb > na + 1 \ge m - 1 + 1$, we have $b > m/n$ too.

Theorem There is a positive number a satisfying $a^2 = 2$.

Write this a as $\sqrt{2}$ or $2^{1/2}$. In general, one can use the same argument to show there is a positive number b satisfying $b^2 = r$ some for any given positive real number r. We shall use $\sqrt{r}$ or $r^{1/2}$ to represent this number b.

Proofs of these results can be found in Text.

So far we know that rational numbers are of the form m/n. But how about irrational numbers? Now we show that every real number has a decimal representation.

The algorithm is: Let a be a positive number. First, find $n_0 \in \mathbb{N}$ such that $0 \le a - n_0 < 1$. The existence of n_0 is guaranteed by Corollary 1 (taking $n = m-1$). Then $0 \le 10(a-a_0) < 10$. Next, we find $n_1 \in \{0, 1, 2, \cdots, 9\}$ to satisfy $n_1 \le 10(a - n_0) < n_1 + 1$. Then $0 \le 10(a - n_0) - n_1 < 1$ and $0 \le 10[10(a - n_0) - n_1] < 10$. We find n_2 such that $n_2 \le 10[10(a - n_0) - n_1] < n_2 + 1$, so $0 \le 10[10(a - n_0) - n_1] - n_2 < 1$ and $0 \le 10\{10[10(a - n_0) - n_1] - n_2\} < 10$. Repeating this process, we get $n_k, k \ge 2$ in $\{0, 1, 2, \cdots, 9\}$, such that

$$0 < a - n_0.n_1n_2\cdots n_k < \frac{1}{10^k}\ , \quad k \ge 1\ .$$

We note

- The sequence $\{n_0, n_0.n_1, n_0.n_1n_2, n_0.n_1n_2n_3, \cdots\}$ is increasing and taking a as its supremum.
- Every positive number has a decimal representation in this way.
- Define T be the mapping from $\mathbb{R}$ to the space of all decimal numbers. The mapping is one-to-one but not onto. A decimal number is not in the range of T iff there is some k such that $n_k = n_{k+1} = n_{k+2} = \cdots = 9$.
- A real number is rational iff its decimal representation becomes periodic after some digit, for instance, $12/5 = 2.4000\cdots$, $80/13 = 0.615384615384\cdots$, etc.

We recall here the notion of a cardinal number, which is needed for further properties of real numbers.

Each nonempty set A is assigned a symbol called its cardinal numbers denoted by $|A|$. It is equal to the number of elements when the set is a finite one. Recall

- Two sets have the same cardinal numbers if there is a one-to-one onto mapping between them.
- Define $|A| \leq |B|$ if there is a one-to-one mapping from A to B, and $|A| < |B|$ if $|A| \leq |B|$ but there is no one-to-one mapping from B to A.
- $\leq$ is a partial order on the sets. That is, (a) $|A| \leq |A|$, (b) $|A| \leq |B|, |B| \leq |C|$ imply $|A| \leq |C|$, and (c) $|A| \leq |B|, |B| \leq |A|$ imply $|A| = |B|$.
- Given two sets A and B, either $|A| < |B|$, $|B| < |A|$ or $|A| = |B|$.

Properties (a) and (b) are obvious, but (c), called Schroder-Bernstein theorem, needs a proof. We will not dig into it.

Denote the cardinal number of the set of natural numbers $\mathbb{N}$ by N_0. A set is called a countable set if it is a finite set or its cardinal number is equal to N_0. An infinite set is uncountable if it is not countable.

Proposition 1. For every infinite set A, $|A| \geq N_0$.

Therefore, countable infinity is the smallest infinity. A little bit surprising is the following result.

Proposition 2. $|\mathbb{Q}| = N_0$.

The proof was done in class. Note that essentially the same argument establishes the following proposition.

Proposition 3. Let $A_j, j = 1, 2, \cdots$, be a sequence of countable sets. Then $\bigcup_{j\geq 1} A_j$ is countable.

Proposition 4. $|\mathbb{R}| > N_0$.

The proof of this proposition makes use of decimal representation. It shows that there is a cardinal number greater than N_0, the smallest one. Therefore, an uncountable set exists. In fact, Cantor shows that it is always true that the cardinal number of the power set of a set is always greater than the cardinal number of the set itself. By taking power of power of power etc, one concludes that there are infinitely many cardinal numbers, one greater than the other.

All these propositions were proved in class, and are not reproduced here.

Since $|\mathbb{N}| < |\mathbb{R}|$, it is natural to ask, is there a set A whose cardinal number lying strictly between the set of natural numbers and the set of real numbers, that is, $|\mathbb{N}| < |A| < |\mathbb{R}|$? This is called the continuum hypothesis. It is known that it is independent of ZF set axioms, the foundation of mathematics used today. It means that the continuum hypotheses can never be proved or disproved!