MATH2050C Assignment 13

No need to hand in any problem.

Supplementary Problems

1. Show that for $x > 0$, the sequence $\{a_n\}, a_n = (1+x/n)^n$ is strictly increasing and bounded from above by $\sum_{k=0}^{\infty} x^k/k!$.
2. Show that for each $m \geq 1, E(x) \geq \sum_{k=0}^{m} x^k/k!$ and conclude $E(x) = \sum_{k=0}^{\infty} x^k/k!$.
3. Show that for $x < 0, E(x) = \lim_{n\to\infty} a_n$ exists and $E(x)E(-x) = 1$.
4. Show that for $x > 0, a, b \in \mathbb{R}$, $x^a x^b = x^{a+b}$ and $(xy)^a = x^a y^a$.

See next page

The Exponential Function and Powers

We study the exponential function, and its inverse function namely the logarithmic function. Then we use it to define the power functions. In the notes in Assignment 4 a preliminary study was present. Here we recall the facts:

1. For $x > 0$, the sequence $\{a_n\}$,
$$a_n = \left(1+\frac{x}{n}\right)^n, \ n \geq 1 ,$$
is strictly increasing and bounded from above. Hence it is convergent and we denote its limit by $E(x)$.
2. $\{a_n\}$ is also convergent for $x \leq 0$ and $E(x)E(-x) = 1$.
3. For each $m \geq 1$,
$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^m}{m!} \leq E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

I leave the proofs of these facts for you in the supplementary problems.

We start with the basic functional relation for $E(x)$.

Theorem 1 For $x, y \in \mathbb{R}$, $E(x+y) = E(x)E(y)$.

Proof Assume $x, y \geq 0$ first.
$$\begin{aligned}\left(1+\frac{x}{n}\right)\left(1+\frac{y}{n}\right) &= \left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right) \\ &= \frac{\left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)}{\left(1+\frac{x+y}{n}\right)} \times \left(1+\frac{x+y}{n}\right) .\end{aligned}$$

Using $1 \leq \frac{1+a+b}{1+a} \leq 1+b$ for $a, b \geq 0$, we have
$$1 \leq \left[\frac{\left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)}{\left(1+\frac{x+y}{n}\right)}\right]^n \leq \left[\left(1+\frac{xy}{n^2}\right)\right]^n .$$

As a subsequence of $(1+xy/n)^n$, $(1+xy/n^2)^{n^2} \to E(xy)$ as $n \to \infty$. For all sufficiently large n, $E(xy)/2 \leq (1+xy/n^2)^{n^2} \leq 2E(xy)$ and
$$(E(xy)/2)^{1/n} \leq (1+xy/n^2)^n \leq (2E(xy))^{1/n} .$$

Using $a^{1/n} \to 1$ as $n \to \infty$, we conclude by Squeeze Theorem that
$$\lim_{n\to\infty} \left(1+\frac{xy}{n^2}\right)^{1/n} = 1,$$

and then

$$\lim_{n\to\infty}\left[\frac{\left(1+\dfrac{x+y}{n}+\dfrac{xy}{n^2}\right)}{\left(1+\dfrac{x+y}{n}\right)}\right]^n = 1 \ .$$

Therefore, by passing limit in

$$\left(1+\frac{x}{n}\right)^n\left(1+\frac{y}{n}\right)^n = \left[\frac{\left(1+\dfrac{x+y}{n}+\dfrac{xy}{n^2}\right)}{\left(1+\dfrac{x+y}{n}\right)}\right]^n \times \left(1+\frac{x+y}{n}\right)^n \ ,$$

we conclude $E(x)E(y) = E(x+y)$ for $x, y \geq 0$. The case of general x, y can be deduced from this relation with the help from Theorem 1. The remaining cases are (a) $x > 0, y < 0$ and $x+y > 0$, (b) $x > 0, y < 0, x+y < 0$, and (b) $x, y, x+y < 0$. For (a), $E(-y)E(x+y) = E(x)$ holds. Thus $E^{-1}(y)E(x+y) = E(x)$ which is $E(x+y) = E(x)E(y)$. Cases (b) and (c) can be proved in a similar way.

Theorem 2 $E(x)$ is strictly increasing, continuous on $\mathbb{R}$. Moreover, $\lim_{n\to\infty} E(x) = \infty$ and $\lim_{n\to-\infty} E(x) = 0$.

Proof Using Fact 3 above, $E(x) > 1 + x > 1$. For $y > x > 0$, $E(y) = E(x+y-x) = E(x)E(y-x) > E(x)$ since $E(y-x) > 1$. Using the relation $E(x) = 1/E(-x)$, one sees that E is also increasing on $(-\infty, 0]$. We conclude that E is strictly increasing on $\mathbb{R}$.

Next, we claim that E is continuous at $x = 0$. For $x \in [0, 1]$,

$$E(x) - 1 = x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right) \leq E(1)x.$$

Therefore, $\lim_{x\to 0^+} E(x) = 1$. On the other hand, $\lim_{x\to 0^-} E(x) = (\lim_{y\to 0^+} E(y))^{-1} = 1/E(0) = 1, y = -x$. Hence E is continuous at $x = 0$. We claim that E is continuous at an arbitrary x_0. Writing $x = x_0 + h$, $E(x) = E(x_0 + h) = E(x_0)E(h) \to E(x_0)E(0) = E(x_0)$ as $h \to 0$.

Finally, from $E(x) \geq 1 + x$, we see that $E(x)$ diverges as $x \to \infty$. On the other hand, using $E(x) = (E(-x))^{-1}$, $E(x)$ decays to 0 as $x \to -\infty$.

Now we relate $E(x)$ to e^x. Recall that it was proved that for every positive a and $q \geq 1$, there is a unique positive number, denoted by $a^{1/q}$, satisfying $(a^{1/q})^q = a$. Given a rational number $r = p/q, p \in \mathbb{Z}, q \geq 1$, define $a^r = (a^{1/q})^p$. It is readily checked that if $r = p'/q'$ then $a^{p'/q'} = a^{p/q}$ hence the (rational) power is well-defined. One can show that $a^{p/q} = (a^p)^{1/q}$ holds. (See Theorem 5.6.7 in Textbook.)

In the following let $E(1) = e = 2.718\cdots$.

Theorem 3 $E(x) = e^x$ for any rational x.

Proof For $q \in \mathbb{N}, E(1) = E(q/q) = E(1/q)^q$, that is the q-power of $E(1/q)$ is the number $E(1)$. According to the definition of the root, $E(1/q) = E(1)^{1/q} = e^{1/q}$. Then for $p > 0$,

$$E(p/q) = E(1/q \cdots + 1/q) = E(1/q)^p = (e^{1/q})^p = e^{p/q} \ .$$

When $p < 0$, $E(p/q) = 1/E(-p/q) = 1/e^{-p/q} = e^{p/q}$.

In view of Theorem 3, it is natural to define the arbitrary power of e to be

$$e^x \equiv E(x) \ .$$

From Theorem 2 and Theorem 5.6.5 in Textbook, the exponential function $E(x)$ has an inverse called the logarithmic function $\ln x$ which is continuous and strictly increasing from $(0,\infty)$ to $\mathbb{R}$. We have $E(\ln x) = x$ for all $x \in (0,\infty)$.

We use the logarithmic function to define the power functions. Indeed, for $x > 0$ and $a \in \mathbb{R}$, define the a-th power of x by

$$x^a \equiv e^{a \ln x} \ .$$

We verify this definition is consistent with the old one when a is a rational number.

Theorem 6 For $x > 0$, $x^{p/q} = e^{p/q \ln x}$.

Proof We have $x = E(\ln x) = E(\ln x/q + \cdots + \ln x/q) = E(\ln x/q)^q$. It means $x^{1/q} = E(\ln x/q)$. Therefore,

$$x^{p/q} \equiv (x^{1/q})^p = E(\ln x/q)^p = E(p/q \ln x) = e^{p/q \ln x} \ .$$

When $p/q > 0$, it is easily seen that $\lim_{x\to 0^+} x^{p/q} = 0$. So $x^{p/q}$ extends to be a continuous function on $[0,\infty)$ by setting its value to be 0 at 0.