## MATH2050C Assignment 12

**Deadline:** April 16, 2024.

**Hand in:** 5.4 no. 3, 4, 7; 5.5 no 3.

**Section 5.4** no. 3, 4, 6, 7, 8, 10, 15. **Section 5.6** no 3,4.

### Supplementary Problems

1. Let function $f$ on $E$ satisfy the condition: There is some constant $C$ and $\alpha > 0$ such that $|f(x) - f(x_0)| \leq C|x - x_0|^\alpha$ for all $x, x_0 \in E$. (It is called Lipschitz continuous when $\alpha = 1$.) Show that $f$ is uniformly continuous on $E$.

2. Let $f$ be a uniformly continuous function on $[0, \infty)$. Show that there is a constant $C$ such that $|f(x)| \leq C(1 + x)$.

3. (Optional) Order the rational numbers in $(0, 1)$ into a sequence $\{x_k\}$. Define a function on $(0, 1)$ by $\varphi(x) = \sum 1/2^k$ where the summation is over all indices $k$ such that $x_k < x$. Show that

   (a) $\varphi$ is strictly increasing and $\lim_{x \to 1^-} \varphi(x) = 1$.

   (b) $\varphi$ is discontinuous at each $x_k$.

   (c) $\varphi$ is continuous at each irrational number in $(0, 1)$.

**See next page**

**Uniform Continuity and Oscillation of Functions**

Let $f$ be continuous on some nonempty set $E$ in $\mathbb{R}$. When $f$ is continuous at some $x_0 \in E$, it means for each $\varepsilon > 0$, there is some $\delta$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in E, |x - x_0| < \varepsilon$. Here $\delta$ in general depending on $x_0$ and $\varepsilon$. Now, f is said to be *uniformly continuous* on $E$ if for each $\varepsilon > 0$, there is a $\delta$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in E, |x - y| < \delta$. If we fix $y$, we immediately see that $f$ is continuous at $y$. Hence a uniformly continuous function on $E$ is continuous on $E$, but the converse is not always true. The function $1/x, \sin 1/x$ are continuous but not uniformly continuous on $(0, 1]$.

**Theorem 1** A uniformly continuous function on a bounded set is bounded.

This theorem holds in all dimensions.

**Proof** Let $f$ be uniformly continuous on the bounded set $E$. Fix $[a, b]$ so that $E \subset [a, b]$. Taking $\varepsilon = 1$, there is $\delta$ such that $|f(x) - f(y)| < 1$ whenever $x, y \in E, |x - y| < \delta$. We chop up $[a, b]$ into finitely many subintervals of length $\delta/2$ and tag them $I_j$'s, $j = 1, \cdots, N$. For those subintervals satisfying $I_j \cap E \neq \phi$, pick a point $x_j$. Then for other $x \in I_j \cap E, |x - x_j| \leq \delta/2 < \delta$, $|f(x) - f(x_j)| < 1$, or $|f(x) \leq |f(x_j)| + 1$. It follows that $|f(x)| \leq \max_j\{|f(x_j) + 1\}$ for all $x \in E$.

**Theorem 2** Every continuous function on $[a, b]$ is uniformly continuous.

We refer to the textbook for a proof. Note that the same proof works for all dimensions where the theorem states as, every continuous function on a closed, bounded set in $\mathbb{R}^n$ is uniformly continuous.

**Example 1** The function $1/x^t, t > 0$, is unbounded on $(0, 1]$. Hence by Theorem 1 it cannot be uniformly continuous on $(0, 1]$. However, by Theorem 2 it is uniformly continuous on $[a, 1]$ for any $a > 0$.

Let $E$ be a nonempty set in $\mathbb{R}$ and $f$ a bounded function on $E$. The *oscillation of $f$ over $E$* is defined to be

$$\mathrm{osc}_E f = \sup_E f - \inf_E f = \sup_{x,y \in E} |f(x) - f(y)|.$$

**Theorem 3** A bounded function $f$ is uniformly continuous on a set $E$ if and only if, given $\varepsilon > 0$, there is some $\delta$ such that on every (open or closed) interval $I$ of length $\delta$, $\mathrm{osc}_{I \cap E} f \leq \varepsilon$.

**Proof** When $f$ is uniformly continuous, for each $\varepsilon > 0$, there is some $\delta$ such that $|f(x) - f(y)| < \varepsilon, x, y \in E, |x - y| < \delta$. Hence when $x, y \in I \cap E$ where the open interval $I$ has length $\delta$, $|x - y| < \delta$ and $|f(x) - f(y)| < \varepsilon$. Hence, taking sup over all $x, y \in I \cap E$, we conclude $\mathrm{osc}_{I \cap E} f \leq \varepsilon$. Conversely, taking $\varepsilon/2 > 0$, there is some $\delta$ such that $\mathrm{osc}_{I \cap E} f \leq \varepsilon/2$ whenever $I$ if of length $\delta$. When $x, y$ satisfy $|x - y| < \delta$, we can find such an interval $I$ containing $x, y$. Therefore, $|f(x) - f(y)| \leq \mathrm{osc}_{I \cap E} f \leq \varepsilon/2 < \varepsilon$.

**Example 2** The function $\sin 1/x$ is not uniformly continuous on $(0, 1]$. Why? Let look at the subinterval $I = (0, \delta)$. No matter how small $\delta > 0$ is, $\mathrm{osc}_I f = 2$. By Theorem 3 (taking $\varepsilon < 2$) it cannot be uniformly continuous.

**Example 3** The function $f(x) = x^2$ is not uniformly on $[0, \infty)$. Let us look at a subinterval of the form $I = (x_0, x_0 + \delta)$. Since this function is increasing $\text{osc}_I f = (x_0 + \delta)^2 - x_0^2 = 2\delta x_0 + 4\delta^2$ which tends to infinity as $x_0 \to \infty$. By Theorem 3, it cannot be uniformly continuous on $[0, \infty)$.

## Monotone Functions

A function is increasing (resp. decreasing ) on an interval $I$ if $f(x) \le f(y)$ (resp. $f(x) \ge f(y)$) whenever $x < y$ in $I$. It is strictly increasing(resp. strictly decreasing) if $f(x) < f(y)$ (resp. $f(x) > f(y)$) whenever $x < y$ in $I$. It is clear that $f$ is increasing (resp. strictly increasing) if and only if $-f$ is decreasing (resp. strictly decreasing).

**Theorem 4** Let $f$ be monotone on the interval $I$ and $c$ an interior point of $I$. Then the right and left limits always exist at $c$.

See textbook for a proof. Consequently a monotone function is continuous at $c$ if and only if $\lim_{x\to c^-} f = \lim_{x\to c^+} f$. (Since $f$ is monotone, $f(c)$ is pinched between the two one-sided limits. Hence $f(c) = \lim_{x\to c^-} f$.) If $f$ is defined at the left endpoint $a$, then $\lim_{x\to a^+} f$ exists and $f$ is continuous at $a$ if and only if $\lim_{x\to a^+} f = f(a)$. A similar situation holds at the right endpoint.

**Theorem 5** The discontinuity set of a monotone function is countable.

**Proof** Let's us assume $f$ is increasing on $[a, b]$. For $c \in (a, b)$, define the jump of $f$ at $c$ to be $j_f(c) = \lim_{x\to c^+} f - \lim_{x\to c^-} f$. Then $j_f(c) > 0$ iff $c$ is a point of discontinuity of $f$. Let $D$ be the set of discontinuity of $f$ in $(a, b)$. We have the decomposition $D = \bigcup_k D_k$ where $D_k = \{x \in (a, b) : j_f(x) \ge 1/k\}$. We claim: Each $D_k$ contains not more than $k(f(b) - f(a))$ many points. Since the countable union of a finite set is countable, $D$ is countable.

Let $c_1 > c_2 > \cdots > c_N$ be points in $(a, b)$. In the following we take $N = 2$ for simplicity. We have

$$\begin{aligned} f(b) - f(a) &= f(b) - \lim_{x\to c_1^+} f + \lim_{x\to c_1^+} f - \lim_{x\to c_1^-} f + \lim_{x\to c_1^-} f - f(a) \\ &= f(b) - \lim_{x\to c_1^+} f + j_f(c_1) + \lim_{x\to c_1^-} f - f(a) \\ &= (f(b) - \lim_{x\to c_1^+} f) + j_f(c_1) + (\lim_{x\to c_1^-} f - \lim_{x\to c_2^+} f) + j_f(c_2) + (\lim_{x\to c_2^-} f - f(a)) \\ &\ge j_f(c_1) + j_f(c_2) \ , \end{aligned}$$

since the three terms in brackets are non-negative. In general, we have

$$f(b) - f(a) \ge \sum_{i=1}^{N} j_f(c_i) \ .$$

Now, if we have $N$ many points in $D_k$, $f(b) - f(a) \ge \sum_{i=1}^{N} j_f(c_i) \ge \sum_{i=1}^{N} 1/k = N/k$, hence $N \le k(f(b) - f(a))$.

The discontinuity set of $f$ on $[a, b]$ is $D$ and possibly including the endpoints, so it is countable. Now, if $f$ is defined on $(a, b)$. Observing $(a, b) = \bigcup_j [a + 1/j, b - 1/j]$, its discontinuity set in $(a, b)$ is also countable since the discontinuity set restricted to each $[a+1/j, b-1/j]$ is countable.