MATH2050C Assignment 11

Deadline: April 9, 2024.

Hand in: 5.3 no. 1, 5, 6, 12, 15.

Section 5.3 no. 1, 3, 4, 5, 6, 12, 13, 15, 17.

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Continuous Functions in $\mathbb{R}^n$

Four theorems are proved in Section 5.4 in our textbook. They are Boundedness Theorem, Max-Min Theorem, Root Theorem and Bolzano Theorem. We would like to extend them to higher dimensions. First recall some facts:

A function f in a set E in $\mathbb{R}^n$ is continuous at $\mathbf{x} \in E$ if for each $\varepsilon > 0$, there is some δ such that $|f(\mathbf{y}) - f(\mathbf{x})| < \varepsilon$ for all $\mathbf{y} \in E, |\mathbf{y} - \mathbf{x}| < \delta$ (here $|\mathbf{x} - \mathbf{y}|$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$.) A vector-valued function F from $E \subset \mathbb{R}^n$ to $\mathbb{R}^m$ is continuous at $\mathbf{x}$ if each component $F_1, \cdots, F_m$ is continuous in E (here $F(\mathbf{x}) = (F_1(\mathbf{x}), \cdots, F_m(\mathbf{x}))$.) Note that Sequential Criterion and Composition Rule hold for these functions.

Boundedness Theorem Let f be continuous on a closed, bounded set E in $\mathbb{R}^n$. It is bounded.

Max-min Theorem Let f be continuous on a closed, bounded set E in $\mathbb{R}^n$. There exist $\mathbf{x}_1$ and $\mathbf{x}_2$ in E such that $f(\mathbf{x}_1) \leq f(\mathbf{x})$ and $f(\mathbf{x}) \leq f(\mathbf{x}_2)$ for all $\mathbf{x} \in E$.

Here we replace $[a, b]$ by a closed, bounded set. A set is closed if it contains all its cluster points. It is bounded if there exists some M such that $|\mathbf{x}| \leq M$ for all $\mathbf{x}$ in this set. Both theorems can be proved by the same arguments as in the one-dimensional case.

Root Theorem Let f be continuous on a connected set E in $\mathbb{R}^n$. Suppose there are $\mathbf{x}_1$ and $\mathbf{x}_2$ in E satisfying $f(\mathbf{x}_1)f(\mathbf{x}_2) < 0$. Then there is some $\mathbf{z} \in E$ such that $f(\mathbf{z}) = 0$.

A set E is connected if every two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in E can be connected by a continuous curve from $\mathbf{x}_1$ to $\mathbf{x}_2$ in E, that is, $\gamma(0) = \mathbf{x}_1, \gamma(1) = \mathbf{x}_2$. Here a continuous curve $\gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$ is map from $[0, 1]$ to E where all components γ_k's are continuous.

The proof is a reduction to one-dimension. Fix a continuous curve γ in E connecting $\mathbf{x}_1$ to $\mathbf{x}_2$. Consider the composite function $g(t) = f(\gamma(t))$ which is a continuous function on $[0, 1]$ to $\mathbb{R}$. As $g(0)g(1) = f(\mathbf{x}_1)f(\mathbf{x}_2) < 0$, By Root Theorem, $g(c) = 0$ for some $c \in (0, 1)$. Thus $f(\xi) = 0$ where $\xi = \gamma(c)$.

Bolzano Theorem Let f be a continuous function in a closed, bounded, connected set E in $\mathbb{R}^n$. For any c lying between $\min f$ and $\max f$, there is some $\mathbf{x} \in E$ such that $f(\mathbf{x}) = c$.

The proof of this theorem is left to you.

In Advanced Calculus we study integration on regions. A region consists of all points lying inside a closed curve (or bounded by several closed curves) as well as all the boundary points (that is, points on these curves). It is a closed, bounded, connected set.