MATH 2050 L8 - Limit Theorems

(Reference: Bartle § 3.2)

Questions

① $\lim(x_n)$ exists ?

② How to compute $\lim(x_n)$?

independent of n!

Def^n: (x_n) is **bounded** if $\exists\, M > 0$ s.t.

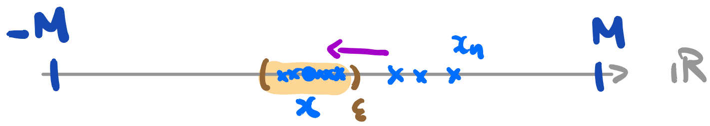
$$|x_n| \leq M \quad \forall\, n \in \mathbb{N}$$

Remark: This is equivalent to $\{x_n : n \in \mathbb{N}\}$ is bounded (as a set).

Thm: (x_n) convergent $\Rightarrow$ (x_n) bounded

Reason:

$(x_n) \to x$



Proof: Since (x_n) is convergent, by def^n, $\exists\, x \in \mathbb{R}$ st.

$$\lim(x_n) = x.$$

[ie $\forall\, \varepsilon > 0$, $\exists\, K \in \mathbb{N}$ st
$|x_n - x| < \varepsilon \quad \forall\, n \geq K$]

Take $\varepsilon = 1$, by def^n of limit, $\exists\, K \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon = 1 \qquad \forall\, n \geq K$$

By Triangle ineq., $\forall n \geq K$.

$$|x_n| = |(x_n - x) + x| \leq |x_n - x| + |x| < 1 + |x|$$

Choose $M := \max\{|x_1|, |x_2|, \ldots, |x_{K-1}|, 1 + |x|\} > 0$

Then. $|x_n| \leq M \quad \forall n \in \mathbb{N}$

$\square$

Remark: The converse of this thm can be used to prove that a sequence is divergent.

i.e. (x_n) unbdd $\Rightarrow$ (x_n) divergent.

Example: $(x_n) := (n)$ unbdd, hence divergent.

Caution: (x_n) bdd $\not\Rightarrow$ (x_n) convergent

(we'll return to this later.)

Recall: $\mathbb{R}$ is a complete ordered field.

(kind of) compatible

Limit Theorems: Suppose $\lim(x_n) = x$, $\lim(y_n) = y$. Then.

(i) $\lim(x_n \pm y_n) = x \pm y$

(ii) $\lim(x_n y_n) = xy$

(iii) $\lim\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$. provided $y_n \neq 0 \ \forall n \in \mathbb{N}$ and $y \neq 0$

(i.e. the limits exist & are equal to the "expected" value.)

Proof: (i) Let $\varepsilon > 0$ be fixed but arbitrary.

Since $\lim(x_n) = x$ and $\lim(y_n) = y$,

$\exists\, K_1, K_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2} \quad \forall n \geq K_1$$

and
$$|y_n - y| < \frac{\varepsilon}{2} \quad \forall n \geq K_2$$

Goal: Find K s.t. $\underbrace{|(x_n+y_n) - (x+y)|}_{\leq} < \varepsilon$

$\underbrace{|(x_n - x)|}_{\substack{\text{small} \\ \because x_n \to x}} + \underbrace{|(y_n - y)|}_{\substack{\text{small} \\ \because y_n \to y}}$

Choose $K := \max\{K_1, K_2\} \in \mathbb{N}$, then $\forall n \geq K$, we have

$$|(x_n \pm y_n) - (x \pm y)| \overset{\because \Delta\text{-ineq.}}{\leq} |x_n - x| + |y_n - y|$$
$$\overset{\because K \geq K_1 \,\&\, K_2}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(ii) Let $\varepsilon > 0$ be fixed but arbitrary.

Since (y_n) is convergent, by previous thm.,

(y_n) is bdd, ie $\exists\, M > 0$ s.t.

$$|y_n| \leq M \quad \forall n \in \mathbb{N}$$

Take $M' := \max\{M, 1 + |x|\} > 0$

By def^{n} of limit (taking $\varepsilon = \varepsilon/_{2M'} > 0$)

then $\exists\, K_1, K_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2M'} \quad \forall n \geq K_1$$

and
$$|y_n - y| < \frac{\varepsilon}{2M'} \quad \forall n \geq K_2$$

Choose $K := \max\{K_1, K_2\} \in \mathbb{N}$, then $\forall n \geq K$, we have

Goal: Find K s.t. $\forall n \geq K$

$|x_n y_n - xy| < \varepsilon$

Estimate:

$|x_n y_n - xy|$

$= |x_n y_n - x y_n + x y_n - xy|$

$\leq |y_n(x_n - x)| + |x(y_n - y)|$

$= \underbrace{|y_n|}_{\substack{\text{bdd} \\ \text{by Thm}}}\ \underbrace{|x_n - x|}_{\text{small}} + \underbrace{|x|}_{\substack{? \\ \text{fixed} \\ \text{number}}}\ \underbrace{|y_n - y|}_{\text{small}}$

$\leq M\, \underbrace{|x_n - x|}_{\varepsilon/2M} + M\, \underbrace{|y_n - y|}_{\varepsilon/2M}$

$$|x_n y_n - xy| = |\underbrace{x_n y_n - x y_n} + \underbrace{x y_n - xy}|$$

$$= |y_n(x_n - x) + x(y_n - y)|$$

$$\leq |y_n|\,|x_n - x| + |x|\,|y_n - y|$$

$$\leq M\,|x_n - x| + |x|\,|y_n - y|$$

$$\leq M'\,|x_n - x| + M'\,|y_n - y|$$

$$< M' \cdot \frac{\varepsilon}{2M'} + M' \cdot \frac{\varepsilon}{2M'} = \varepsilon$$

(iii) Since $\left(\frac{x_n}{y_n}\right) = \left(x_n \cdot \frac{1}{y_n}\right)$, using (ii), it suffices to show

(#): $\lim\left(\frac{1}{y_n}\right) = \frac{1}{y}$ provided $y_n \neq 0 \ \forall n \in \mathbb{N}$ and $y \neq 0$.

Let $\varepsilon > 0$ be fixed but arbitrary.

We first establish a lemma.

Lemma: $\exists \tilde{K} \in \mathbb{N}$ st

$$|y_n| \geq \frac{|y|}{2} \qquad \forall n \geq \tilde{K}$$

Pf of Lemma: Since $\lim(y_n) = y$,

by take $\varepsilon := \frac{|y|}{2} > 0$ since $y \neq 0$.

$\exists \tilde{K} \in \mathbb{N}$ st $|y_n - y| < \varepsilon = \frac{|y|}{2}$

$\forall n \geq \tilde{K}$

By reverse triangle ineq, $\forall n \geq \tilde{K}$.

$$|y_n| = |y + (y_n - y)| \geq \big||y| - |y_n - y|\big|$$

$$\geq |y| - \frac{|y|}{2} = \frac{|y|}{2}.$$

Goal: Find K st $\forall n \geq K$

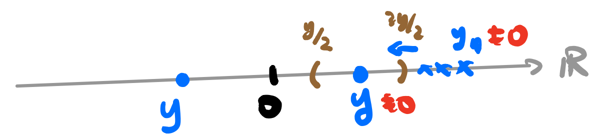
$$\left|\frac{1}{y_n} - \frac{1}{y}\right| < \varepsilon$$

Estimate: (WANT: $|y_n - y|$)

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right|$$

$$= \frac{|y_n - y|}{|y_n||y|}$$

$$= \underbrace{\frac{1}{|y_n|}}_{????} \ \underbrace{\frac{1}{|y|}}_{\text{fixed no.}} \ \underbrace{|y_n - y|}_{\text{small}}$$



Lemma: $|y_n| \geq \frac{|y|}{2} > 0 \quad \forall n \geq \tilde{K}$

Since $\lim(y_n) = y$, taking $\varepsilon / \frac{2}{|y|^2} > 0$.

$\exists\, K' \in \mathbb{N}$ s.t. $|y_n - y| < \varepsilon / \frac{2}{|y|^2} \quad \forall n \geq K'$ ⟵ (*)

Choose $K := \max\{\tilde{K}, K'\} \in \mathbb{N}$. then $\forall n \geq K$, we have

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \left| \frac{y_n - y}{y_n y} \right| = \frac{1}{|y_n|} \cdot \frac{1}{|y|} \, |y_n - y|$$

$$\leq \underbrace{\boxed{\frac{1}{|y|/2}}}_{\text{by Lemma}} \cdot \frac{1}{|y|} \cdot \underbrace{\varepsilon / \frac{2}{|y|^2}}_{\substack{(*) \\ \text{choice of } K'}} = \varepsilon$$

$\square$

Non-example: The assumptions in (iii) are necessary.

Consider $(y_n) := \left(\frac{1}{n}\right) \to y = 0$. then

$\left(\frac{1}{y_n}\right) = (n)$ is divergent.

Remark: Converse of the Thm is not true.

Eg. $\underbrace{(x_n) = \left(\frac{1}{n}\right)}_{\substack{\text{convergent} \\ \text{to } 0}}$, $\underbrace{(y_n) = (n)}_{\text{divergent}}$ then $\underbrace{(x_n y_n) = (1)}_{\text{convergent}}$

Thm: Let $(x_n), (y_n)$ be two sequences of real numbers s.t.

$$x_n \leq y_n \quad \forall n \in \mathbb{N} \qquad \text{(**)}$$

THEN, $\lim(x_n) \leq \lim(y_n)$ provided that their limits exist.

Remarks: (i) For $(**)$, it is also sufficient to have

$$x_n \le y_n \quad \forall n \ge L \quad \text{for some fixed } L.$$

(ii) Even if we assume $x_n < y_n \ \forall n \in \mathbb{N}$ in $(**)$,

we still get $\lim(x_n) \le \lim(y_n)$ only.

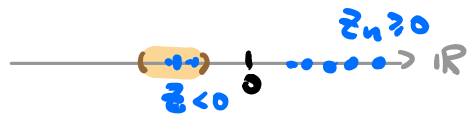
E.g.) $0 < \frac{1}{n} \quad \forall n \in \mathbb{N}$ But $\lim(0) = 0 = \lim(\frac{1}{n})$.

Proof of Thm: By Limit Thm (i), it suffices to show

$(*)$: $\underset{\text{conv. seq.}}{(z_n)}$ s.t. $z_n \ge 0 \ \forall n \in \mathbb{N} \Rightarrow \lim(z_n) =: z \ge 0$.

Suppose NOT., then $z := \lim(z_n) < 0$.

Take $\varepsilon = \frac{|z|}{2} > 0$, then $\exists K \in \mathbb{N}$ s.t.



$$|z_n - z| < \varepsilon = \frac{|z|}{2} \quad \forall n \ge K.$$

$$\Rightarrow \quad z_n < z + \frac{|z|}{2} = -\frac{|z|}{2} < 0 \quad \forall n \ge K.$$

Contradicting $z_n \ge 0 \ \forall n \in \mathbb{N}$!

$\blacksquare$

Summary: "Limit Thm" ASSUME $\lim(x_n), \lim(y_n)$ exist.

$$\begin{cases} \lim(x_n \pm y_n) = \lim(x_n) \pm \lim(y_n) \\ \lim(x_n y_n) = \lim(x_n) \cdot \lim(y_n) \\ \lim\left(\frac{x_n}{y_n}\right)_{\neq 0} = \frac{\lim(x_n)}{\lim(y_n) \neq 0} \end{cases}$$

If $x_n \le y_n$ $\forall n \in \mathbb{N}$

then $\lim(x_n) \le \lim(y_n)$

also OK. $\forall n \ge K$ for some $K \in \mathbb{N}$

Q: How to prove that $\lim(x_n)$ exist?

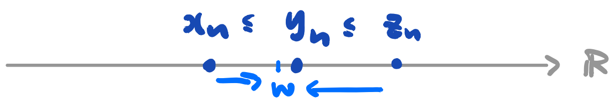
Thm: ("Squeeze / Sandwich Theorem")

Let $(x_n), (y_n), (z_n)$ be seq. of real numbers s.t.

(1) $x_n \leq y_n \leq z_n \quad \forall n \in \mathbb{N}$ ($\forall n \geq K$ for some K)

(2) $\lim(x_n) = w = \lim(z_n)$

THEN, $\lim(y_n) = w$.



Remark: We do <u>NOT</u> need to assume $\lim(y_n)$ exists, it follows from the theorem.

E.g.) $\lim\left(\frac{\sin n}{n}\right) = 0$ because $\underset{\to 0}{-\frac{1}{n}} \leq \underset{\to 0}{\frac{\sin n}{n}} \leq \underset{\to 0}{\frac{1}{n}}$

Proof: Let $\varepsilon > 0$ be fixed but arbitrary.

$\lim(x_n) = w \Rightarrow \exists K_1 \in \mathbb{N}$ s.t. $|x_n - w| < \varepsilon \quad \forall n \geq K_1$ —— (*)

$\lim(z_n) = w \Rightarrow \exists K_2 \in \mathbb{N}$ s.t. $|z_n - w| < \varepsilon \quad \forall n \geq K_2$ —— (**)

Choose $K := \max\{K_1, K_2\}$, then $\forall n \geq K$

$$-\varepsilon \overset{(*)}{<} x_n - w \overset{(1)}{\leq} y_n - w \overset{(1)}{\leq} z_n - w \overset{(**)}{<} \varepsilon$$

i.e. $|y_n - w| < \varepsilon$

$\square$

Thm: ("Ratio Test")

Let (x_n) be a seq. s.t.

(1) $x_n > 0 \quad \forall n \in \mathbb{N}$

(2) $\lim \left(\frac{x_{n+1}}{x_n}\right) = L < 1$

crucial!

THEN, $\lim (x_n) = 0$.

Motivation

Geometric seq.

$(ar^n) \to 0$

provided $|r| < 1$

Ex: Prove this!

E.g.) Consider $(x_n) = \left(\frac{n}{2^n}\right)$, then

$$\left(\frac{x_{n+1}}{x_n}\right) = \left(\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right) = \left(\frac{n+1}{n} \cdot \frac{1}{2}\right) \to \frac{1}{2} < 1$$

By Ratio Test. $\lim \left(\frac{n}{2^n}\right) = 0$.

Proof: Idea: Compare (x_n) with a geometric seq. (b^n), where $0<b<1$ and apply Squeeze Thm!

$\frac{x_{n+1}}{x_n} \approx L < 1$

$x_{n+1} \approx L x_n$

$x_{n+2} \approx L^2 x_n$

$x_{n+k} \approx L^k x_n$

Since $L < 1$, $\exists\, r \in \mathbb{R}$ s.t. $L < r < 1$.

Take $\varepsilon = r - L > 0$, by (2), $\exists\, K \in \mathbb{N}$ s.t.

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon = r - L \qquad \forall n \geq K$$

$$\Rightarrow \quad 0 \overset{(1)}{<} \frac{x_{n+1}}{x_n} < L + (r - L) = r < 1$$

Thus, $x_{n+1} < r x_n \quad \forall n \geq K$.

i.e $\quad 0 < x_n < r x_{n-1} < r^2 x_{n-2} < \cdots < r^{n-K} x_K$

K fixed

Note: $\lim\limits_{n\to\infty} (r^{n-K} x_K) = 0$ since $r < 1$.

By Sandwich Thm, $\lim (x_n) = 0$ $\blacksquare$

Remark: Ratio Test fails if $L = 1$.

Consider the seq. $(x_n) = (n)$, which is divergent

But $\left(\frac{x_{n+1}}{x_n}\right) = \left(\frac{n+1}{n}\right) \longrightarrow 1 = 1$

Ex: Construct an example that $\left(\frac{x_{n+1}}{x_n}\right) \to 1$ from below.