MATH 2050 - Absolute value and some inequalities

(Reference: Bartle § 2.2)

Def^{n}: (Absolute value) Let $a \in \mathbb{R}$.

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Note: $|a| \geq 0 \quad \forall a \in \mathbb{R}$

Prop: (a) $|ab| = |a| \cdot |b|$

(b) $|a|^2 = a^2$

*(c) Let $c \geq 0$. Then $|a| \leq c \iff -c \leq a \leq c$

(d) $-|a| \leq a \leq |a|$

Proof: (a) We exhaust all possible cases from Trichotomy (O2).

Case 1: Either a or b is 0.

Then, $ab = 0 \Rightarrow |ab| = 0$.

Also, if $a = 0$, then $|a| = 0 \Rightarrow |a| \cdot |b| = 0$.

Same for $b = 0$. So, $|ab| = |a||b| = 0$.

Case 2: $a > 0$ and $b < 0$.

Then, by Prop. last time, $ab < 0 \Rightarrow |ab| = -ab$.

Also, $a > 0 \Rightarrow |a| = a$, $b < 0 \Rightarrow |b| = -b$ $\Big\}$ $|a| \cdot |b| = a \cdot (-b) = -ab$. \\ Same

Ex: Check this!

Case 3: $a > 0$ and $b > 0$

Case 4: $a < 0$ and $b < 0$

) left as exercise

Case 5: $a < 0$ and $b > 0$ (same as case 2)

(b) Take $b = a$ in (a),

$$a^2 = |a^2| = |ab| = |a||b| = |a|\cdot|a| = |a|^2.$$

$\because a^2 \geq 0 \quad \forall a \in \mathbb{R}$.

(c) Exhaust all cases of a by trichotomy (Exercise)

(d) Follows from (c) by taking $c = |a| \geq 0$. $\blacksquare$

Some Useful Inequalities

(1) AM-GM inequality: $\sqrt{\underline{ab}_{\geq 0}} \leq \frac{1}{2}(a+b) \quad \forall a, b \geq 0$

(2) Triangle inequality: $|a+b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$

(3) Bernoulli's inequality: $(\underline{1+x}_{\geq 0})^n \geq 1 + n\cdot x \quad \forall x > -1, \forall n \in \mathbb{N}$

Proof: (1) Let $a, b \geq 0$, then $\sqrt{a}, \sqrt{b}$ exist (Assume this).

By previous lemma,

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = \underbrace{(\sqrt{a})^2}_{\text{def}^{\underline{n}}} - 2\sqrt{a}\sqrt{b} + \underbrace{(\sqrt{b})^2}_{\text{def}^{\underline{n}}}$$

$$= a - 2\sqrt{a}\sqrt{b} + b$$

Rearranging gives the desired inequality.

(2) By (d) above, we have

$$\left.\begin{aligned} -|a| \leq a \leq |a| \\ -|b| \leq b \leq |b| \end{aligned}\right\} \overset{\text{add}}{\Longrightarrow} -(|a|+|b|) \leq a+b \leq |a|+|b|$$

$$\overset{(c)}{\Longrightarrow} \quad |a+b| \leq |a| + |b|.$$

(3) Induction on n.

$\underline{n=1}$: Trivial since $(1+x)^n = 1+x = 1+n\cdot x$. when $n=1$.

Assume $\underline{n=k \text{ is true}}$, then for $n = k+1$,

$$(1+x)^{k+1} = \underbrace{(1+x)}_{\substack{\because x>-1 \\ \geq 0}} \underbrace{(1+x)^k}_{\text{IV } n=k}$$

$$\geq (1+x)(1+k\cdot x) \qquad \left(\because\ n=k \text{ is true and } x>-1\right)$$

$$= 1+(k+1)x + k\cdot x^2$$

$$\geq 1+(k+1)x \qquad \left(\because\ k>0,\ x^2\geq 0\right)$$

By M.I., we are done. $\square$

Remark: Let $a, b \geq 0$. Then

$$a \leq b \iff a^2 \leq b^2 \iff \sqrt{a} \leq \sqrt{b}.$$

Prop: (Reversed Triangle Ineq.)

$$\big||a|-|b|\big| \leq |a-b| \qquad \forall a,b \in \mathbb{R}.$$

Pf: Tutorial.