MATH 2050 - Uniform Continuity

(Reference: Bartle § 5.4)

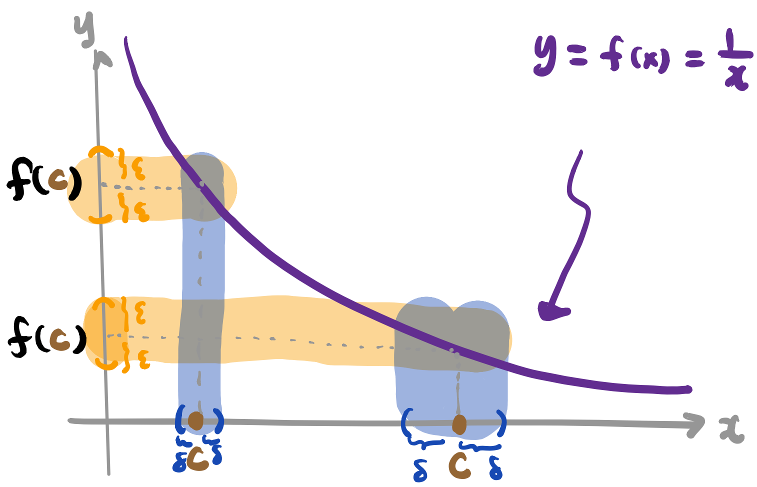
Recall: Let $f: A \to \mathbb{R}$.

- f is cts at $c \in A$ $\iff$ $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.
  $|f(x) - f(c)| < \varepsilon \quad \forall\, |x - c| < \delta$

- f is cts on A $\overset{\text{def}^{\text{n}}}{\iff}$ f is cts at EVERY $c \in A$
  $\iff \forall c \in A, \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, c) > 0,$
  s.t. $|f(x) - f(c)| < \varepsilon \quad \forall\, |x - c| < \delta$

Caution: The choice of δ depends on BOTH ε AND c.

Example: $f: (0, \infty) \to \mathbb{R}$ $\quad f(x) := \frac{1}{x}$ $\quad$ cts on $(0, \infty)$



FOR THE SAME $\varepsilon > 0$

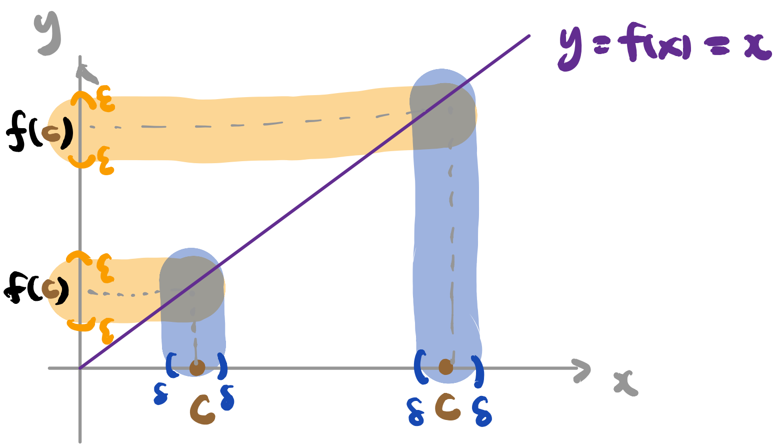
If $c \approx 0$, then we need to choose a much smaller δ s.t.

$|f(x) - f(c)| < \varepsilon \quad \forall\, |x - c| < \delta$

Idea: This function is NOT "uniformly" cts

$\because$ δ is NOT "uniform" in c

Example: $f:(0,\infty)\to\mathbb{R}$ $\quad f(x):=x$ cts on $(0,\infty)$



FOR THE SAME $\varepsilon>0$

You can choose ONE $\delta>0$

s.t. it works for ALL $c\in A$

$|f(x)-f(c)|<\varepsilon \quad \forall\ |x-c|<\delta$

Idea: This function is "uniformly" cts.

Def^{n}: $f:A\to\mathbb{R}$ is uniformly continuous (on A)

iff $\forall\,\varepsilon>0,\ \exists\,\delta=\delta(\varepsilon)>0$ s.t.

$$|f(u)-f(v)|<\varepsilon \quad \forall\ u,v\in A,\ |u-v|<\delta$$

Remark: (1) uniform cts $\Rightarrow$ cts on A ($\because$ take $v=c\in A$)

(2) Uniform continuity is a "global" concept. It does NOT make sense to talk about uniform continuity at one point $c\in A$.

Q: How to see if $f:A\to\mathbb{R}$ is uniformly cts (on A)?

We first begin with a "non-uniform continuity" criteria.

Prop: $f: A \to \mathbb{R}$ is <u>NOT</u> uniformly continuous

$\iff \exists\, \varepsilon_0 > 0$ s.t. $\forall\, \delta > 0$, $\exists\, u_\delta, v_\delta \in A$

s.t. $|u_\delta - v_\delta| < \delta$ <u>BUT</u> $|f(u_\delta) - f(v_\delta)| \geq \varepsilon_0$

$\iff \exists\, \varepsilon_0 > 0$ and seq. $(u_n), (v_n)$ in A

s.t. $|u_n - v_n| < \frac{1}{n}$ <u>BUT</u> $|f(u_n) - f(v_n)| \geq \varepsilon_0$ $\forall n \in \mathbb{N}$

Proof: Take negation of def^{n} and choose $\delta = \frac{1}{n}$.

$\blacksquare$

Example: Show that $f: (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is <u>NOT</u> uniformly continuous on $(0,\infty)$.

Proof: Take $(u_n) := \left(\frac{1}{n}\right)$ and $(v_n) := \left(\frac{1}{n+1}\right)$ in $(0,\infty)$.

THEN, $|u_n - v_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n}$ $\forall n \in \mathbb{N}$

<u>BUT</u> $|f(u_n) - f(v_n)| = |n - (n+1)| = 1 \geq \varepsilon_0 := \frac{1}{2} > 0$.

By Prop, f is <u>NOT</u> uniformly cts on $(0,\infty)$.

$\blacksquare$

Exercise: Show that $f: [a,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is uniformly cts on $[a,\infty)$ for any fixed $a > 0$.

Idea: We can say more about "uniform continuity" of $f: A \to \mathbb{R}$ if A is an interval.

- Uniform Continuity Thm
- Continuous Extension Thm

Uniform Continuity Thm

closed & bdd interval.

$f : [a,b] \to \mathbb{R}$ cts on $[a,b]$ $\Rightarrow$ f is uniformly cts on $[a,b]$.

Proof: Argue by contradiction. Suppose NOT, i.e. f is NOT uniformly cts. Then, by non-uniform continuity criteria, $\exists\ \varepsilon_0 > 0$ and seq. $(u_n), (v_n)$ in $[a,b]$

(*) .... [s.t. $|u_n - v_n| < \frac{1}{n}$ BUT $|f(u_n) - f(v_n)| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$]

By Bolzano-Weierstrass Thm, since (u_n) is bdd.

$\Rightarrow \exists$ subseq. (u_{n_k}) of (u_n) s.t.

$$\lim_{k\to\infty} (u_{n_k}) = x^* \in [a,b]$$

Claim: $\lim_{k\to\infty} (v_{n_k}) = x^*$

Pf: $|u_{n_k} - v_{n_k}| < \frac{1}{n_k} \quad \forall k \in \mathbb{N}$ $\xRightarrow{\text{as } k\to\infty}$ $\lim_{k\to\infty} (v_{n_k}) = x^*$ by limit thm.

By continuity of f at $x^* \in [a,b]$.

$$0 < \varepsilon_0 \overset{(*)}{\leq} \lim_{k\to\infty} |f(u_{n_k}) - f(v_{n_k})| = |f(x^*) - f(x^*)| = 0$$

Contradiction!

$\square$

Q: When can we extend a cts $f : (a,b) \to \mathbb{R}$ to a cts function $\bar{f} : [a,b] \to \mathbb{R}$?

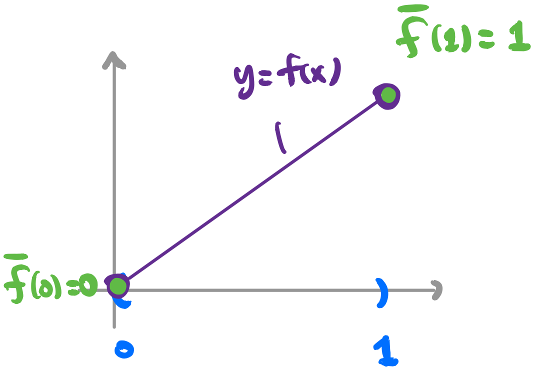
(s.t. $\bar{f}(x) = f(x) \quad \forall x \in (a,b)$.)

Examples:

Yes!

$f(x) = x$ on $x \in (0,1)$

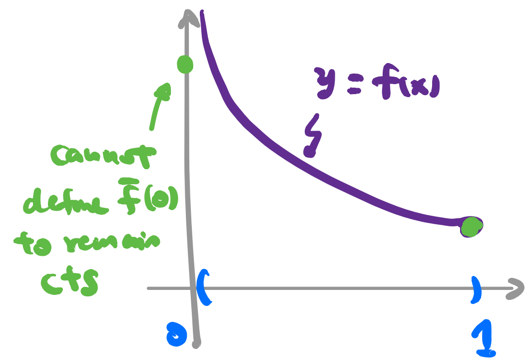
extend $\rightsquigarrow$ $\bar{f}(x) = x$ on $x \in [0,1]$



No!

$f(x) = \frac{1}{x}$ on $x \in (0,1)$

$\rightsquigarrow$ $\nexists$ cts extension $\bar{f}$ to $[0,1]$.



Continuous Extension Thm

If $f : (a,b) \to \mathbb{R}$ is uniformly cts (on (a,b))

then $\exists$ an "extension" $\bar{f} : [a,b] \to \mathbb{R}$ s.t.

(i) $\bar{f}(x) = f(x) \quad \forall x \in [a,b]$

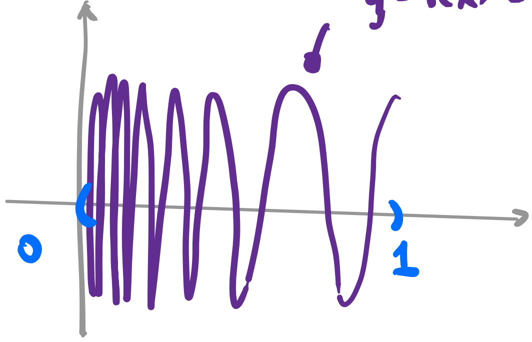
(ii) $\bar{f}$ is cts on $[a,b]$

Remarks: (a) $\bar{f}$ is uniformly cts by Uniform Continuity Thm

(b) Such an extension $\bar{f}$ is unique.

Example:

$y = f(x) = \sin \frac{1}{x}$ remain bdd on $(0,1)$



BUT not unif. cts on $(0,1)$

[Ex: Prove this.]

We will use the following lemma in the proof.

Lemma: Let $f: A \to \mathbb{R}$ be uniformly cts. THEN,

$$(x_n) \text{ Cauchy seq in } A \implies (f(x_n)) \text{ Cauchy seq in } \mathbb{R}$$

Proof of Lemma: Let $\varepsilon > 0$.

By uniform continuity of f, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t.

(#)... $\left[\ |f(u) - f(v)| < \varepsilon \quad \text{whenever } u, v \in A \text{ st } |u - v| < \delta\ \right\}$

Let (x_n) be a Cauchy seq in A. By ε-H def^{n}, for this $\delta > 0$ above, $\exists\ H = H(\delta) \in \mathbb{N}$ s.t.

$$|x_m - x_n| < \delta \quad \forall\, n, m \geq H$$

By (#), $|f(x_m) - f(x_n)| < \varepsilon \quad \forall\, n, m \geq H$

So, $(f(x_n))$ is Cauchy.

$\square$

Proof of Continuous Extension Thm:

It suffices to show the existence of $\lim_{x \to a} f(x)$, $\lim_{x \to b} f(x)$, then we can define $\bar{f}: [a, b] \to \mathbb{R}$ as

$$\bar{f}(x) := \begin{cases} f(x), & x \in (a, b) \\ \lim_{x \to a} f(x), & x = a \\ \lim_{x \to b} f(x), & x = b \end{cases}$$

Claim: $\lim_{x \to a} f(x)$ exists.

Pf: By Sequential Criteria, it suffices to prove that

$\exists L \in \mathbb{R}$ s.t. for ANY seq. (x_n) in (a,b) s.t.

$\lim(x_n) = a$ we have $\lim(f(x_n)) = L$

Step 1: Find one such L.

Choose $x_n := a + \frac{1}{n}$ $\forall n \in \mathbb{N}$ (defined when n is large)

Note: $(x_n) \to a$ hence is Cauchy

By Lemma, $(f(x_n))$ is Cauchy, hence converging to some $L \in \mathbb{R}$.

Step 2: Show that the L we obtained in Step 1 works for ALL seq. $(x_n') \to a$ ((x_n') in (a,b)).

Take an arbitrary seq (x_n') in (a,b) converging to a.

[Idea: $\underbrace{x_n \approx x_n'}_{\text{close}} \overset{\text{unif.}}{\underset{\text{cts}}{\Longrightarrow}} \underbrace{f(x_n) \approx f(x_n')}_{\text{close}}$]

Since $\lim(x_n) = a = \lim(x_n')$, we have

$$\lim |x_n - x_n'| = 0 \qquad \text{by Limit theorem}$$

To see $(f(x_n')) \to L$. Suppose, by Step 1, $(f(x_n')) \to L'$

Let $\varepsilon > 0$. By uniformly continuity of f, $\exists\, \delta = \delta(\varepsilon) > 0$

(*)...... s.t. $|f(u) - f(v)| < \varepsilon$ when $u, v \in (a,b)$, $|u - v| < \delta$

Now, $\lim |x_n - x_n'| = 0 \Rightarrow \exists\, K = K(\delta) \in \mathbb{N}$ s.t.

$$|x_n - x_n'| < \delta \quad \forall n \geq K$$

Hence, we have from (*),

$$|f(x_n) - f(x_n')| < \varepsilon \quad \forall n \geq K$$

Take $n \to \infty$, we obtain $|L - L'| \leq \varepsilon$ but $\varepsilon > 0$ is arbitrary. Then, we have $L = L'$.

$\square$

Picture: