MATH 2050B Mathematical Analysis I
2023-24 Term 1
Problem Set 6
due on Oct 27, 2023 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.** All the exercises below are taken from the textbook.

Required Readings: Chapter 3.4

Optional Readings: none

Problems to hand in

Section 3.4: Exercise # 5, 7(b), 10, 14, 19

Suggested Exercises

Section 3.4: Exercise # 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 15, 16, 18

Challenging Exercises (optional)

1. Let (x_n) be the sequence of real numbers defined for $n \in \mathbb{N}$ by (with the convention that $0! = 1$)

$$x_n := \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

   (a) Prove that (x_n) converges to some real number $e \in \mathbb{R}$.

(b) Show that $e = \lim \left(1 + \frac{1}{n}\right)^n$.

(c) Prove that e is irrational.

2. This is a continuation of Challenging Exercise 2 of Problem Set 5.

   (a) Find a sequence (x_n) of positive real numbers with $\limsup(x_n) = +\infty$ such that $\lim(s_n) = 0$.

   (b) Let (y_n) be the sequence defined by $y_n := x_{n+1} - x_n$, $n \in \mathbb{N}$. Show that for $n \geq 2$,

$$x_n - s_n = \frac{1}{n} \sum_{k=1}^{n-1} k y_k.$$

   Suppose that $\lim(n y_n) = 0$ and that (s_n) is convergent. Prove that (x_n) is convergent.