MATH2050A Mathematical Analysis I

Suggested solution to HW 6

(1) Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function such that $f([0,1]) \subset \mathbb{Q}$. Show that f is a constant function.

Solution. Suppose f is not a constant function. Then there exists $x_1, x_2 \in [0,1]$ such that $f(x_1) < f(x_2)$. By the density of irrational numbers, we can find $k \in \mathbb{R}\backslash\mathbb{Q}$ such that $f(x_1) < k < f(x_2)$. Since $f : [0,1] \to \mathbb{R}$ is a continuous function, it follows from Bolzano's Intermediate Value Theorem that there exists $c \in [0,1]$ between x_1 and x_2 such that $f(c) = k$. This contradicts the assumption that $f([0,1]) \subset \mathbb{Q}$. □

(2) Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ is a function given by $f(x) = \sup\{x^2, \cos x\}$. Show that there exists an absolute minimum point $x_0 \in [0, \frac{\pi}{2}]$ for f. Moreover, show that x_0 is the solution to $x^2 = \cos x$.

Solution. Recall that $\sup\{a, b\} = \max\{a, b\} = \frac{a+b+|a-b|}{2}$. Since x^2 and $\cos x$ are both continuous on $[0, \frac{\pi}{2}]$, it follows from Theorem 5.2.2 and 5.2.4 that f is also continuous on $[0, \frac{\pi}{2}]$. By the Maximum-Minimum Theorem, there exists an absolute minimum point $x_0 \in [0, \frac{\pi}{2}]$ for f.

Observe that $x_0 \neq 0, \frac{\pi}{2}$ since $f(0) = 1, f(\frac{\pi}{2}) = \frac{\pi^2}{4} > f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

Suppose $x_0^2 \neq \cos x_0$. Then either $x_0^2 > \cos x_0$ or $x_0^2 < \cos x_0$.

Case 1: Suppose $x_0^2 > \cos x_0$. Then, by the continuity of $x^2 - \cos x$, we can find x_1 such that $0 < x_1 < x_0$ and $x_1^2 > \cos x_1$. Now, since x^2 is increasing on $[0, \frac{\pi}{2}]$, we have

$$f(x_1) = x_1^2 < x_0^2 = f(x_0).$$

This contradicts the fact that x_0 is an absolute minimum point for f on $[0, \frac{\pi}{2}]$.

Case 2: Suppose $x_0^2 < \cos x_0$. By the same argument in Case 1, we can find x_2 such that $x_0 < x_2 < \frac{\pi}{2}$ and $x_2^2 < \cos x_2$. Now, since $\cos x$ is decreasing on $[0, \frac{\pi}{2}]$, we have

$$f(x_2) = \cos x_2 < \cos x_0 = f(x_0).$$

This again contradicts the fact that x_0 is an absolute minimum point for f on $[0, \frac{\pi}{2}]$.

Therefore x_0 is a solution to $x^2 = \cos x$. □

(3) Show that the function $f(x) = x^{-2}$ is uniformly continuous on $[1, +\infty)$ but is not on $(0, +\infty)$.

Solution. (i) Note that, for $x, u \geq 1$,

$$|f(x) - f(u)| = \left| \frac{(x+u)(x-u)}{x^2 u^2} \right| \leq \frac{|x| + |u|}{|x|^2|u|^2}|x - u| = \left(\frac{1}{|x||u|^2} + \frac{1}{|x|^2|u|} \right) |x - u|$$
$$\leq 2|x - u|.$$

Let $\varepsilon > 0$. Take $\delta = \varepsilon/2$. Now, if $x, u \in [1, +\infty)$ and $|x - u| < \delta$, then

$$|f(x) - f(u)| \leq 2|x - u| < 2\delta = \varepsilon.$$

Hence f is uniformly continuous on $[1, +\infty)$.

(ii) Let $\varepsilon_0 = 1$. Consider the sequences $\{x_n\} = \{1/n\}$ and $\{u_n\} = \{1/(n+1)\}$. Then $\{x_n\}$ and $\{u_n\}$ both lie in $(0, +\infty)$, and satisfy $\lim_{n\to+\infty}(u_n - x_n) = 0$. However,

$$f(u_n) - f(x_n) = (n+1)^2 - n^2 = 2n + 1 \geq 1 = \varepsilon_0 \qquad \text{for all } n \in \mathbb{N}.$$

By Nonuniform Continuity Criteria, f is not uniformly continuous on $(0, +\infty)$.

□

(4) Suppose $f, g : A \to \mathbb{R}$ are uniformly continuous and bounded on A, show that fg is also uniformly continuous. Is the same assertion true without the boundedness assumption? Justify your answer.

Solution. (i) Since f and g are bounded on A, there exists $M > 0$, such that $|f|, |g| \leq M$ on A. Let $\varepsilon > 0$. Since f and g are both uniformly continuous on A, there exists $\delta > 0$ such that if $x, u \in A$ and $|x - u| < \delta$, then

$$|f(x) - f(u)| < \frac{\varepsilon}{2M} \quad \text{and} \quad |g(x) - g(u)| < \frac{\varepsilon}{2M}.$$

Hence, if $x, u \in A$ and $|x - u| < \delta$, then

$$\begin{aligned}
|fg(x) - fg(u)| &= |f(x)g(x) - f(x)g(u) + f(x)g(u) - f(u)g(u)| \\
&\leq |f(x)g(x) - f(x)g(u)| + |f(x)g(u) - f(u)g(u)| \\
&= |f(x)|\,|g(x) - g(u)| + |g(u)|\,|f(x) - f(u)| \\
&\leq M\,|g(x) - g(u)| + M\,|f(x) - f(u)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Therefore, fg is uniformly continuous on A.

(ii) The same assertion is not true without the Boundedness assumption. For example, consider $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = g(x) = x$.
Given any $\varepsilon > 0$, if we choose $\delta = \varepsilon$, then whenever $x, u \in \mathbb{R}$ with $|x - u| < \delta$, we have

$$|f(x) - f(u)| = |g(x) - g(u)| = |x - u| < \delta = \varepsilon.$$

Hence, f and g are uniformly continuous on $\mathbb{R}$.
Consider the sequences $\{x_n\} = \{n\}$, $\{u_n\} = \{n + \frac{1}{n}\}$. Then $\lim_{n\to+\infty}(u_n - x_n) = 0$ but

$$|fg(u_n) - fg(x_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} \geq 2 \qquad \text{for all } n \in \mathbb{N}.$$

By Nonuniform Continuity Criteria, fg is not uniformly continuous on $\mathbb{R}$.

□

(5) Show that if f is continuous on $[0, +\infty)$ and is uniformly continuous on $[a, +\infty)$ for some $a > 0$, then f is uniformly continuous on $[0, +\infty)$.

Solution. Let $\varepsilon > 0$. Since f is uniformly continuous on $[a, \infty)$, there exists $\delta_1 > 0$ such that if $x, u \in [a, \infty)$ and $|x - u| < \delta_1$, then

$$|f(x) - f(u)| < \varepsilon.$$

Since f is continuous on $[0, \infty)$, it is continuous on $[0, a + 1]$ and hence uniformly continuous on $[0, a + 1]$ by Uniform Continuity Theorem 5.4.3. Then there exists $\delta_2 > 0$ such that if $x, u \in [0, a + 1]$ and $|x - u| < \delta_2$, then

$$|f(x) - f(u)| < \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2, 1\}$. Suppose $x, u \in [0, \infty)$ and $|x - u| < \delta$. Then either $x, u \in [a, \infty)$ and $|x - u| < \delta_1$; or $x, u \in [0, a + 1]$ and $|x - u| < \delta_2$. In either case,

$$|f(x) - f(u)| < \varepsilon.$$

Hence f is uniformly continuous on $[0, \infty)$. □