MATH2050A Mathematical Analysis I

Suggested solution to HW 5

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $S = \{x \in \mathbb{R} : f(x) = 0\}$. Show that S is closed in the sense that if $x_n \in S$ and $x_n \to x$, then $x \in S$.

Solution. Let (x_n) be a sequence in S such that $x_n \to x$. For all $n \in \mathbb{N}$, $x_n \in S$ implies that $f(x_n) = 0$. By the Sequential Criterion for Continuity, since (x_n) converges to x, we must have $(f(x_n))$ converges to $f(x)$. Therefore $f(x) = 0$, that is $x \in S$. □

(2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$f(m2^{-n}) = m2^{-m}$$

for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Show that $f(x) = x$ for all $x \in \mathbb{R}$.

Solution. We first show that the set $A := \{m2^{-n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$, that is, given any $x, y \in \mathbb{R}$ with $x < y$, there exists $a \in A$ such that $x < a < y$.

Since $y - x > 0$, it follows from Corollary 2.4.5 that there exists $n \in \mathbb{N}$ such that $1/n < y - x$. As $2^n \geq n$, we have $2^{-n} < y - x$ and so $2^n y - 2^n x > 1$. Then there exists $m \in \mathbb{Z}$ such that $2^n x < m < 2^n y$. Thus, the number $a := m2^{-n} \in A$ satisfies $x < a < y$.

Next, we show that $f(x) = x$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. By the density of A in $\mathbb{R}$, for any $n \in \mathbb{N}$, there exists $x_n \in A$ such that

$$x < x_n < x + \frac{1}{n}.$$

Clearly, $x_n \neq x$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. Since f is continuous at x, it follows from the Sequential Criterion for Continuity that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x.$$

□

(3) Using the ε, δ terminology to show that

(a)

$$\lim_{x\to 2} \sqrt{\frac{2x+1}{x+3}} = 1.$$

(b)

$$\lim_{x\to 1} \frac{x^2 - 3x}{x+3} = \frac{-1}{2}.$$

Solution. (a) First, note that for any $x, y \geq 0$,

$$|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| = |x - y|.$$

Thus, for $x > 0$, we have

$$\left|\sqrt{\frac{2x+1}{x+3}} - 1\right|^2 \leq \left|\frac{2x+1}{x+3} - 1\right| = \frac{|x-2|}{|x+3|}.$$

Let $\varepsilon > 0$. Take $\delta = \min\{\varepsilon^2, 1\}$. Now if $0 < |x - 2| < \delta$, then

$$|x + 3| = |(x - 2) + 5| \geq 5 - |x - 2| \geq 5 - 1 = 4,$$

and so

$$\left|\sqrt{\frac{2x+1}{x+3}} - 1\right| \leq \sqrt{\frac{|x-2|}{|x+3|}} \leq \sqrt{\frac{\varepsilon^2}{4}} = \frac{\varepsilon}{2} < \varepsilon.$$

Therefore $\lim_{x\to 2} \sqrt{\frac{2x+1}{x+3}} = 1$.

(b) For $x \neq -3$,

$$\left|\frac{x^2 - 3x}{x+3} - (\frac{-1}{2})\right| = \left|\frac{2x^2 - 5x + 3}{2(x+3)}\right| = \frac{|2x-3|}{2|x+3|}|x - 1|.$$

Note that, if $|x - 1| < 1$, then

$$|2x - 3| = |2(x - 1) - 1| \leq 2|x - 1| + 1 < 3,$$

and

$$|x + 3| = |(x - 1) + 4| \geq 4 - |x - 1| > 3.$$

Let $\varepsilon > 0$. Take $\delta = \min\{\varepsilon, 1\}$. Now if $0 < |x - 1| < \delta$, then

$$\left|\frac{x^2 - 3x}{x+3} - (\frac{-1}{2})\right| = \frac{|2x-3|}{2|x+3|}|x - 1| < \frac{3}{2 \cdot 3}\delta \leq \varepsilon/2 < \varepsilon.$$

Therefore $\lim_{x\to 1} \frac{x^2 - 3x}{x+3} = \frac{-1}{2}$.

□

(4) Show that the following limit does not exist:

$$\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right).$$

Solution. Denote $f(x) = \sin\left(\frac{1}{x^2}\right)$. Let

$$x_n := \frac{1}{\sqrt{2n\pi}} \text{ and } y_n := \frac{1}{\sqrt{2n\pi + \pi/2}} \quad \text{for } n \in \mathbb{N}.$$

Then (x_n) and (y_n) are non-zero sequences such that

$$\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(y_n) = 0.$$

However, $f(x_n) = \sin(2n\pi) = 0$ for all $n \in \mathbb{N}$, so that $\lim_{n\to\infty} f(x_n) = 0$; while $f(y_n) = \sin(2n\pi + \pi/2) = 1$ for all $n \in \mathbb{N}$, so that $\lim_{n\to\infty} f(y_n) = 1$. In view of the Sequential Criterion for Limits, $\lim_{x\to 0} f(x)$ does not exist. □

(5) If $f : A \to \mathbb{R}_{\geq 0}$ and c is a cluster point of A so that f has a limit $L \geq 0$ at c. Show that $\sqrt{f}$ has limit $\sqrt{L}$ at c.

Solution. Note that for any $x, y \geq 0$,

$$|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| = |x - y|.$$

Let $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon^2 \qquad \text{whenever } x \in A \text{ with } 0 < |x - c| < \delta.$$

Now, if $x \in A$ and $0 < |x - c| < \delta$, then

$$|\sqrt{f(x)} - \sqrt{L}| \leq \sqrt{|f(x) - L|} < \sqrt{\varepsilon^2} = \varepsilon.$$

Hence $\lim_{x\to c} \sqrt{f}(x) = \sqrt{L}$. □