MATH2050A Mathematical Analysis I

Suggested solution to HW 4

(1) Let $x_1 < x_2$ be two given real numbers. Define the sequence inductively by

$$x_n = \frac{1}{3}x_{n-1} + \frac{2}{3}x_{n-2}$$

for all $n > 2$. Show that $\{x_n\}$ is convergent and find the limit.

Solution. Note that

$$x_{n+2} - x_{n+1} = \frac{1}{3}x_{n+1} + \frac{2}{3}x_n - x_{n+1} = -\frac{2}{3}(x_{n+1} - x_n) \qquad \text{for } n \in \mathbb{N}.$$

Thus

$$x_{n+2} - x_{n+1} = -\frac{2}{3}(x_{n+1} - x_n) = (-\frac{2}{3})^2(x_n - x_{n-1}) = \cdots = (-\frac{2}{3})^n(x_2 - x_1).$$

Summing up the expression, we have

$$\begin{aligned}\sum_{k=0}^{n}(x_{k+2} - x_{k+1}) &= (x_2 - x_1)\sum_{k=0}^{n}(-\frac{2}{3})^k \\ x_{n+2} - x_1 &= (x_2 - x_1)\frac{1 - (-\frac{2}{3})^{n+1}}{1 - (-\frac{2}{3})} \\ x_{n+2} &= x_1 + \frac{3}{5}(x_2 - x_1)[1 - (-\frac{2}{3})^{n+1}].\end{aligned}$$

Since $\lim(-\frac{2}{3})^{n+1} = 0$, we have $\lim(x_n) = x_1 + \frac{3}{5}(x_2 - x_1) = \frac{2}{3}x_1 + \frac{3}{5}x_2$. □

(2) If $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{x_n}$ for all $n \geq 1$, show that $\{x_n\}$ is a contractive sequence, i.e. there exists $C \in [0, 1)$ such that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|.$$

Show that $\{x_n\}$ is convergent and find the limit.

Solution. It is easy to check that $x_n \geq 2$ for all $n \in \mathbb{N}$ by induction. Then, for all $n \geq 2$,

$$\begin{aligned}|x_{n+1} - x_n| &= \left|(2 + \frac{1}{x_n}) - (2 + \frac{1}{x_{n-1}})\right| \\ &= \frac{1}{x_n x_{n-1}}|x_n - x_{n-1}| \\ &\leq \frac{1}{4}|x_n - x_{n-1}|.\end{aligned}$$

So $\{x_n\}$ is a contractive sequence, and hence is convergent by Theorem 3.5.8. Suppose $x = \lim(x_n)$. Then we have $x = 2 + \frac{1}{x}$, so that $x^2 - 2x - 1 = 0$. Solving the equation, we obtain $x = 1 + \sqrt{2}$ as the other root $1 - \sqrt{2}$, which is less than 2, is rejected. □

(3) Find an example of sequence $\{x_n\}$ such that it is not a Cauchy sequence but for any fixed $p \in \mathbb{N}$, $x_{n+p} - x_n \to 0$ as $n \to +\infty$.

Solution. Consider the sequence (x_n) defined by $x_n := \sum_{k=1}^{n} \frac{1}{k}$ for $n \in \mathbb{N}$.

Then (x_n) is not a Cauchy sequence since for any $n \in \mathbb{N}$,

$$|x_{2n} - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

On the other hand, for any fixed $p \in \mathbb{N}$,

$$|x_{n+p} - x_n| = \frac{1}{n+1} + \cdots + \frac{1}{n+p} \leq \frac{p}{n+1} \to 0 \quad \text{as } n \to +\infty,$$

and therefore, $\lim(x_{n+p} - x_n) = 0$ by Squeeze Theorem. □

(4) Show that if $x_n > 0$ for all $n \in \mathbb{N}$, then $x_n \to 0$ as $n \to +\infty$ if and only if $x_n^{-1} \to +\infty$ as $n \to +\infty$.

Solution. Suppose $\lim(x_n) = 0$. Then for any $M > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - 0| < 1/M$, and hence $x_n^{-1} = |x_n - 0|^{-1} > M$. Therefore, $x_n^{-1} \to +\infty$ as $n \to +\infty$.

On the other hand, suppose $x_n^{-1} \to +\infty$ as $n \to +\infty$. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n^{-1} > 1/\varepsilon$, and hence $|x_n - 0| = x_n < \varepsilon$. Therefore, $\lim(x_n) = 0$. □