MATH2050A Mathematical Analysis I

Suggested solution to HW 3

(1) Establish the convergence or divergence of the sequence $\{x_n\}_{n=1}^{\infty}$ where

$$x_n = \sum_{k=1}^{n} \frac{1}{n+k}.$$

Solution. The sequence $\{x_n\}_{n=1}^{\infty}$ is increasing since

$$\begin{aligned} x_{n+1} - x_n &= \sum_{k=1}^{n+1} \frac{1}{n+1+k} - \sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=2}^{n+2} \frac{1}{n+k} - \sum_{k=1}^{n} \frac{1}{n+k} \\ &= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} = \frac{1}{(2n+1)(2n+2)} \geq 0. \end{aligned}$$

It is also bounced above since

$$x_n \leq \sum_{k=1}^{n} \frac{1}{n} = 1.$$

By the monotone convergence theorem, $\{x_n\}_{n=1}^{\infty}$ is convergent. □

(2) Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ where

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

is convergent using the monotone convergence theorem.

Solution. The sequence $\{x_n\}_{n=1}^{\infty}$ is clearly increasing. It is also bounded above since

$$x_n = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1 + 1 - \frac{1}{n} \leq 2.$$

By the monotone convergence theorem, $\{x_n\}_{n=1}^{\infty}$ is convergent. □

(3) Suppose $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} (-1)^n x_n$ exists. Show that $\{x_n\}_{n=1}^{\infty}$ is convergent.

Solution. By Theorem 3.2.9, the existence of $x := \lim_{n\to\infty} (-1)^n x_n$ implies that $\lim_{n\to\infty} |(-1)^n x_n| = |x|$. Since $x_n \geq 0$ for all $n \in \mathbb{N}$, we have $|(-1)^n x_n| = x_n$. Therefore $\{x_n\}_{n=1}^{\infty}$ is convergent. □

(4) Show that if $\{x_n\}_{n=1}^{\infty}$ is unbounded, then there exists a sub-sequence $\{x_{n_j}\}_{j=1}^{\infty}$ which is non-zero so that $\frac{1}{x_{n_j}} \to 0$ as $j \to +\infty$.

Solution. As $\{x_n\}_{n=1}^{\infty}$ is unbounded, we have for any $M > 0$, there is $n \in \mathbb{N}$ such that $|x_n| > M$.

Pick $n_1 \in \mathbb{N}$ such that $|x_{n_1}| > 1$.
Then pick $n_2 \in \mathbb{N}$ such that $|x_{n_2}| > \max\{2, |x_1|, |x_2|, \ldots, |x_{n_1}|\}$. So $|1/x_{n_2}| < 1/2$ and $n_2 > n_1$.

Suppose $n_1 < n_2 < \cdots < n_k$ are chosen so that $|1/x_{n_j}| < 1/j$ for $1 \leq j \leq k$.

Pick $n_{k+1} \in \mathbb{N}$ such that $|x_{n_{k+1}}| > \max\{k+1, |x_1|, |x_2|, \ldots, |x_{n_k}|\}$. So $|1/x_{n_{k+1}}| < 1/(k+1)$ and $n_{k+1} > n_k$.
Continue in this way, we obtain a non-zero sub-sequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$|1/x_{n_k}| < 1/k \qquad \text{for all } k \in \mathbb{N}.$$

Now $\lim_{k\to\infty}(1/x_{n_k}) = 0$ follows immediately from the Squeeze Theorem. □

(5) Suppose for every sub-sequence of $\{x_n\}_{n=1}^{\infty}$, there exists a sub-sequence that converges to 0, show that $\{x_n\}_{n=1}^{\infty}$ is convergent with limit 0.

Solution. Suppose on the contrary that $\{x_n\}_{n=1}^{\infty}$ does not converge to 0. Then, by Theorem 3.4.4, there exist $\varepsilon_0 > 0$ and a sub-sequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$|x_{n_k} - 0| \geq \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

By the assumption, $\{x_{n_k}\}_{k=1}^{\infty}$ has a further sub-sequence $\{x_{n_{k_j}}\}_{j=1}^{\infty}$ that converges to 0. This contradicts the fact that $|x_{n_{k_j}} - 0| \geq \varepsilon_0$ for all $j \in \mathbb{N}$.

Therefore $\{x_n\}_{n=1}^{\infty}$ is convergent with limit 0. □