MATH2050A Mathematical Analysis I

Suggested solution to HW 2

(1) Let S be a non-empty subset of $\mathbb{R}$. Show that S is bounded if and only if there exists a closed and bounded interval I such that $S \subseteq I$.

Solution. Suppose S is bounded. Then there exist $a, b \in \mathbb{R}$ such that

$$a \leq x \leq b \quad \text{for all } s \in S.$$

Hence $S \subseteq [a, b]$.

Suppose there exists a closed and bounded interval I such that $S \subseteq I$. Let a, b be the left endpoint and right endpoint of I respectively. Then, for any $s \in S$, we have $a \leq s \leq b$. So S is bounded. □

(2) Let $f, g : S \to \mathbb{R}$ be two real-valued functions. Suppose that $\sup\{f(x)+g(x) : x \in S\}$, $\sup\{f(x) : x \in S\}$ and $\sup\{g(x) : x \in S\}$ exist in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in S\} \leq \sup\{f(x) : x \in S\} + \sup\{g(x) : x \in S\}.$$

Show that in general $\leq$ cannot be replaced by $=$ by providing a counter-example.

Solution. Denote $a \coloneqq \sup\{f(x) : x \in S\}$ and $b \coloneqq \sup\{g(x) : x \in S\}$. For any $x \in S$, we have $f(x) \leq a$ and $g(x) \leq b$, and hence $f(x) + g(x) \leq a + b$. So $a + b$ is an upper bound of the set $\{f(x) + g(x) : x \in S\}$,which implies that

$$\sup\{f(x) + g(x) : x \in S\} \leq \sup\{f(x) : x \in S\} + \sup\{g(x) : x \in S\}.$$

Consider $S = [-1, 1]$ and $f(x) = -g(x) = x$ for any $x \in S$. Then $f(x) + g(x) = 0$ for any $x \in S$, and so $\sup\{f(x) + g(x) : x \in S\} = 0$. However, $\sup\{f(x) : x \in S\} = \sup\{g(x) : x \in S\} = 1$, and thus $\sup\{f(x) : x \in S\} + \sup\{g(x) : x \in S\} = 2$. □

(3) (a) Let $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for all $n \in \mathbb{N}$. Determine whether $\{x_n\}$ is convergent or not.

(b) Let $x_1 = 1$ and $x_{n+1} = \sqrt{2x_n}$ for all $n \in \mathbb{N}$, show that $\{x_n\}$ is convergent.

Solution. (a) Since $x_1 > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \in \mathbb{N}$, it follows easily from induction that $x_n > 0$ for all $n \in \mathbb{N}$. Thus

$$x_{n+1}x_n = x_n^2 + 1 \quad \text{for all } n \in \mathbb{N}.$$

Suppose $\{x_n\}$ is convergent and the limit is $x \in \mathbb{R}$. By letting $n \to \infty$ in the above equation, we have $x^2 = x^2+1$, and hence $0 = 1$, which is absurd. Therefore, $\{x_n\}$ must be divergent.

(b) Let $P(n)$ be the statement that $x_n \leq 2$ and $x_{n+1} \geq x_n$. Clearly $P(1)$ is true since $x_1 = 1 < \sqrt{2} = x_2$. Suppose $P(n)$ is true. Then

$$x_{n+1} = \sqrt{2x_n} \leq \sqrt{2 \cdot 2} = 2,$$

and

$$x_{n+2} = \sqrt{2x_{n+1}} \geq \sqrt{2x_n} = x_{n+1}.$$

So $P(n+1)$ is true. By Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.
Since $\{x_n\}$ is increasing and bounded above, it follows from Monotone Convergence Theorem that $\{x_n\}$ is convergent.
By letting $x = \lim(x_n)$ and solving the equation $x^2 = 2x$, we find that $x = 2$ since $x = 0$ is impossible ($x_n \geq x_1 = 1$ for all $n \in \mathbb{N}$.)

□

(4) (a) Show that $\left\{\dfrac{n+3}{n^3 - 2n + 4}\right\}_{n=1}^{\infty}$ is convergent.

(b) Show that $\left\{(5n^6)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ is convergent.

Solution. (a) For $n \in \mathbb{N}$, write

$$\frac{n+3}{n^3 - 2n + 4} = \frac{\frac{1}{n^2} + \frac{3}{n^3}}{1 - \frac{2}{n} + \frac{4}{n^3}}.$$

By Theorem 3.2.3(a), we have

$$\lim\left(\frac{1}{n^2} + \frac{3}{n^3}\right) = 0 + 0 = 0,$$

and

$$\lim\left(1 - \frac{2}{n} + \frac{4}{n^3}\right) = 1 - 0 + 0 = 1 \neq 0.$$

So, by Theorem 3.2.3(b), we have

$$\lim\left(\frac{\frac{1}{n^2} + \frac{3}{n^3}}{1 - \frac{2}{n} + \frac{4}{n^3}}\right) = \frac{0}{1} = 0.$$

Therefore, $\left\{\dfrac{n+3}{n^3 - 2n + 4}\right\}_{n=1}^{\infty}$ converges to 0.
As a demonstration, we will also show the convergence of the sequence by definition of limit.
Note that, for $n \geq 2$, we have $n^3 - 2n \geq \frac{1}{2}n^3$ and hence

$$\left|\frac{n+3}{n^3 - 2n + 4} - 0\right| = \frac{n+3}{n^3 - 2n + 4} \leq \frac{n+3}{\frac{1}{2}n^3 + 4} \leq \frac{n+3n}{\frac{1}{2}n^3} = \frac{8}{n^2} \leq \frac{8}{n}.$$

Let $\varepsilon > 0$. By Archimedean property, there is $N \in \mathbb{N}$ such that $N > \max\{2, \frac{8}{\varepsilon}\}$.
Now, for $n \geq N$, we have

$$\left|\frac{n+3}{n^3 - 2n + 4} - 0\right| \leq \frac{8}{n} \leq \frac{8}{N} < \varepsilon.$$

(b) For $n \in \mathbb{N}$, write
$$(5n^6)^{\frac{1}{n}} = 5^{\frac{1}{n}} \cdot \left(n^{\frac{1}{n}}\right)^6.$$
By Examples 3.1.11, we have $\lim(5^{\frac{1}{n}}) = 1$ and $\lim(n^{\frac{1}{n}}) = 1$.
So, by Theorem 3.2.3(a), we have
$$\lim\left(5^{\frac{1}{n}} \cdot \left(n^{\frac{1}{n}}\right)^6\right) = 1 \cdot (1)^6 = 1.$$
Therefore, $\left\{(5n^6)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ converges to 1.
Again we will also show the convergence of the sequence by definition of limit.
Let $x_n = (5n^6)^{\frac{1}{n}} - 1 \geq 0$ for $n \in \mathbb{N}$. Then, for $n \geq 12$, we have
$$5n^6 = (1+x_n)^n = \sum_{k=0}^{n} C_k^n x_n^k \geq C_7^n x_n^7 = \frac{n(n-1)\cdots(n-6)}{7!}x_n^7 \geq \frac{(n/2)^7}{7!}x_n^7,$$
and hence
$$0 \leq x_n^7 \leq \frac{5 \cdot 7! \cdot 2^7}{n}.$$
Let $\varepsilon > 0$. By Archimedean property, there is $N \in \mathbb{N}$ such that $N > \max\{12, \frac{5 \cdot 7! \cdot 2^7}{\varepsilon^7}\}$.
Now, for $n \geq N$, we have
$$\left|(5n^6)^{\frac{1}{n}} - 1\right| = x_n \leq \sqrt[7]{\frac{5 \cdot 7! \cdot 2^7}{n}} \leq \sqrt[7]{\frac{5 \cdot 7! \cdot 2^7}{N}} < \varepsilon.$$

□