MATH2050A Mathematical Analysis I

Suggested solution to HW 1

(1) Using the Axioms to show that for all $a, b \in \mathbb{R}$,

$$(-a)^2 = a^2 \quad \text{and} \quad (a + (-b))^2 = a^2 + (-2ab) + b^2.$$

Solution. First we show that if $a + b = 0$, then $b = -a$ $(*)$. Indeed,

$$\begin{aligned}
b &= b + 0 && \text{(by A3)}\\
&= b + (a + (-a)) && \text{(by A4)}\\
&= (a + b) + (-a) && \text{(by A1, A2)}\\
&= 0 + (-a) && \text{(by assumption)}\\
&= -a && \text{(by A3).}
\end{aligned}$$

Thus, we have $-a = (-1)a$ $(**)$ because

$$\begin{aligned}
a + (-1)a &= 1 \cdot a + (-1)a && \text{(by M3)}\\
&= (1 + (-1)) \cdot a && \text{(by D)}\\
&= 0 \cdot a && \text{(by A4)}\\
&= a \cdot 0 && \text{(by M1)}\\
&= 0 && \text{(by Theorem 2.1.2(c)).}
\end{aligned}$$

Hence, to show that $(-a)^2 = a^2$, it suffices to show that $(-a)^2 + (-a^2) = 0$. Now

$$\begin{aligned}
(-a)^2 + (-a^2) &= (-a)^2 + (-1)a^2 && \text{(by } (**)\text{)}\\
&= (-a)^2 + ((-1)a)a && \text{(by M1, M2)}\\
&= (-a)^2 + (-a)a && \text{(by } (**)\text{)}\\
&= (-a)(-a + a) && \text{(by D)}\\
&= (-a) \cdot 0 && \text{(by A4)}\\
&= 0 && \text{(by Theorem 2.1.2(c)).}
\end{aligned}$$

For the second equality,

$$\begin{aligned}
(a + (-b))^2 &= a(a + (-b)) + (-b)(a + (-b) && \text{(by D)}\\
&= a^2 + a(-b) + (-b)a + (-b)^2 && \text{(by D)}\\
&= a^2 + a(-b) + a(-b) + b^2 && \text{(by M1, first equality)}\\
&= a^2 + a((-1)b) + a((-1)b) + b^2 && \text{(by } (**)\text{)}\\
&= a^2 + (-1)(ab) + (-1)(ab) + b^2 && \text{(by M1, M2)}\\
&= a^2 + (-1)(2ab) + b^2 && \text{(by D)}\\
&= a^2 + (-2ab) + b^2 && \text{(by } (**)\text{).}
\end{aligned}$$

$\square$

(2) Show that for all $n \in \mathbb{N}$,

$$(n, n+1) \cap \mathbb{N} = \emptyset.$$

Show further that if $m, n \in \mathbb{Z}$ such that $m < n$, then $m + 1 \leq n$.

Solution. Suppose on the contrary that for some $n_0 \in \mathbb{N}$, $(n_0, n_0+1) \cap \mathbb{N} \neq \emptyset$. Then $S \coloneqq \{n \in \mathbb{N} : n_0 < n < n_0 + 1\}$ is a nonempty subset of $\mathbb{N}$. By the Well-Ordering Property of $\mathbb{N}$, S has a least element m. Now, $n_0 < m < n_0 + 1$, which implies that $0 < m - n_0 < 1$ and so

$$0 < (m - n_0)^2 < m - n_0 < 1.$$

Thus $(m - n_0)^2 + n_0$ is a natural number that satisfies

$$n_0 < (m - n_0)^2 + n_0 < m < n_0 + 1,$$

contradicting the fact m is the least element of S.

Suppose there exist $m, n \in \mathbb{Z}$ such that $m < n$ but $m + 1 > n$. Then $n - m \in \mathbb{N}$ and satisfies $0 < n - m < 1$. This contradicts the result above. □

(3) Suppose S is a non-empty bounded subset in $\mathbb{R}$. Is $\sup S$ necessarily inside S? Justify your answer.

Solution. Not true. For example $S \coloneqq [0, 1)$ is a non-empty bounded subset of $\mathbb{R}$ but $\sup S = 1 \notin S$. We will check that $\sup S = 1$.

Clearly S is bounded above by 1. Let $0 < \varepsilon < 1$. Then $0 < 1 - \varepsilon < 1 - \varepsilon/2 < 1$. So $1 - \varepsilon$ is not an upper bound of S because $1 - \varepsilon/2 \in S$. Therefore $\sup S = 1$.

□

(4) Show that if A, B are bounded subsets of $\mathbb{R}$, then

$$\sup(A + B) = \sup A + \sup B, \quad \text{and} \quad \inf(A + B) = \inf A + \inf B$$

where $A + B = \{a + b : a \in A, b \in B\}$.

Solution. We further assume that A and B are non-empty. Otherwise, the corresponding suprema and infima do not exist.

We will only prove $\sup(A+B) = \sup A + \sup B$ as the other can be proved similarly.

By the Completeness Axiom of $\mathbb{R}$, both $\sup A$ and $\sup B$ exist. It is clear that $\sup A + \sup B$ is an upper bound of $A + B$. Indeed, for any $a \in A$, $b \in B$, we have $a \leq \sup A$, $b \leq \sup B$ and hence $a + b \leq \sup A + \sup B$.

Next we show that $\sup A + \sup B$ is the least upper bound of $A + B$. Let $\varepsilon > 0$. By Lemma 2.3.4, there are $a_0 \in A$ and $b_0 \in B$ such that $a_0 > \sup A - \frac{\varepsilon}{2}$ and $b_0 > \sup B - \frac{\varepsilon}{2}$. Hence $a_0 + b_0 > \sup A + \sup B - \varepsilon$. By Lemma 2.3.4 again, we have $\sup(A + B) = \sup A + \sup B$. □

(5) Show that $2^n \geq n + 1$ for all $n \in \mathbb{N}$. Show further that for any $x > 0$, there is $n \in \mathbb{N}$ such that $\frac{1}{2^n} < x$.

Solution. We will prove the first assertion by induction. The inequality is true if $n = 1$. If we assume that $2^k \geq k + 1$, then $2^{k+1} \geq 2(k + 1) \geq k + 2$. Thus, if the inequality is true for k, then it also holds for $k+1$. Therefore, Mathematical Induction implies that the inequality is true for all $n \in \mathbb{N}$.

If $x > 0$, then the Archimedean Property implies that there exists $n \in \mathbb{N}$ such that $n \geq \frac{1}{x}$. Hence, by the inequality above, the same $n \in \mathbb{N}$ satisfies

$$\frac{1}{2^n} \leq \frac{1}{n+1} < \frac{1}{n} \leq x.$$

□

(6) Show by using completeness that there is $x \in \mathbb{R} \backslash \mathbb{Q}$ so that $x > 0$ and $x^3 = 2$.

Solution. First, we will show that there is $x \in \mathbb{R}$ such that $x > 0$ and $x^3 = 2$. Let $S = \{s \in \mathbb{R} : s^3 < 2\}$. Then $1 \in S$ and S is bounded above by 2. By the Completeness Axiom of $\mathbb{R}$, $x := \sup S$ exists. Note $x \geq 1 > 0$.

We will make use of the following elementary inequality: if $0 \leq a \leq b$, then

$$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) \leq (b - a)nb^{n-1}.$$

Suppose $x^3 < 2$. Take $\varepsilon = \dfrac{1}{2}\min\{\dfrac{2 - x^3}{3(x + 1)^2}, 1\} > 0$. Then

$$(x + \varepsilon)^3 - x^3 \leq 3\varepsilon(x + \varepsilon)^2 < \frac{(x + \varepsilon)^2}{(x + 1)^2}(2 - x^3) \leq 2 - x^3,$$

and so $(x + \varepsilon)^3 < 2$. Since $x < x + \varepsilon$, this contradicts the fact that $x = \sup S$.

Suppose $x^3 > 2$. Take $\varepsilon = \dfrac{x^3 - 2}{6x^2} > 0$. Then

$$x^3 - (x - \varepsilon)^3 \leq 3\varepsilon x^2 < x^3 - 2,$$

and so $(x - \varepsilon)^3 > 2$. Now $x - \varepsilon$ is an upper bound of S since $s > x - \varepsilon \implies s^3 > (x - \varepsilon)^3 > 2 \implies s \notin S$. Again this contradicts the fact that $x = \sup S$.

Therefore, we must have $x^3 = 2$. It remains to show that such x is irrational. Suppose x is rational and $x = p/q$, where p, q are positive integers that are relatively prime.

Then $p^3 = 2q^3$. This implies that p^3 is even, and so is p. Therefore, since p and q do not have 2 as a common factor, then q must be odd.

Since p is even, then $p = 2m$ for some $m \in \mathbb{N}$, and hence $4m^3 = q^3$. This implies that q^3 is even, and so is q.

Since the hypothesis $x \in \mathbb{Q}$ leads to the contradictory conclusion that q is both odd and even, it must be false. □