Math 2050, intermediate value Theorem

Recall the definition of continuity of a function.

Definition 0.1. *A function $f : A \to \mathbb{R}$ is said to be continuous at $c \in A$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x-c| < \delta, x \in A$, then $|f(x) - f(c)| < \varepsilon$. The function f is said to be continuous on A, if f is continuous at $c \in A$ for all $c \in A$.*

Important remark: the choice of δ is a-priori depending on the point $c \in A$.

Example: $\lim_{x\to c} x^n = c^n$ for any given $n \in \mathbb{N}$.

Proof. We first consider the error:

$$
\begin{aligned}
|f(x) - f(c)| &= |x^n - c^n| \\
&= |x-c| \left| \sum_{k=0}^{n-1} x^{n-1-k} c^k \right| \\
&\leq |x-c| \left(\sum_{k=0}^{n-1} |x|^{n-1-k} |c|^k \right).
\end{aligned} \tag{0.1}
$$

Like before, it suffices to control the "coefficient". We fix $\delta = \min\{\Lambda\varepsilon, 1\}$ where we will specify Λ later. Then if $|x - c| < \delta \leq 1$, we have

$$
|x|^{n-1-k} \leq (|x-c| + |c|)^{n-1-k} \leq (1 + |c|)^{n-1-k}. \tag{0.2}
$$

Hence,

$$
\begin{aligned}
|f(x) - f(c)| &\leq |x-c| \left(\sum_{k=0}^{n-1} (1+|c|)^{n-1-k} |c|^k \right) \\
&= M_c |x-c|
\end{aligned} \tag{0.3}
$$

where M_c is the number depending on the value of c. Then by choosing $\Lambda = M^{-1}$ which also depends on c, we have if $|x-c| < \delta$,

$$
|f(x) - f(c)| < \varepsilon.
$$

In this way, it is clear that the choice of δ is possibly depending also on the given point! This (in)dependence will be important later! □

Some algebra of continuity (using sequence criterion):

Theorem 0.1. *Let $A \subset \mathbb{R}$ and $f, g : A \to \mathbb{R}$ be functions continuous at $c \in A$ and $\lambda \in \mathbb{R}$. Then $f+g, f-g, \lambda f, fg$ are continuous at $c \in A$. If $g(x) \neq 0$ on A, then fg^{-1} is continuous at $c \in A$.*

As a immediate applications: polynomials are continuous on $\mathbb{R}$.

More properties of continuous functions (also using sequence criterion):

Theorem 0.2. *Let $A, B \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ such that $f(A) \subset B$. If f is continuous at $c \in A$ and g is continuous at $f(c)$, then $g \circ f$ is continuous at $c \in A$.*

Proof. Let $x_n \in A$ such that $x_n \to c$. Since f is continuous at c, we have $f(x_n) \to f(c)$. Using sequence criterion again, since g is continuous at $f(c)$ and $f(x_n) \to f(c)$, we have $g\left(f(x_n)\right) \to g(f(c))$. Since x_n is arbitrary, we have the continuity of $g \circ f$ at $c \in A$. $\square$

Example: $f(x) = \sqrt{x + \sqrt{x}}, \sin|x|, \mathit{etc}$ are continuous on $\mathbb{R}^+$.

1. Continuous functions on closed and bounded intervals

Examples:

(1) $f(x) = x^{-1}$ on $(0, 1]$;
(2) $f(x) = (x + 1)^{-1}$ on $[0, 1]$.

If we allow the interval to be open, the first example states that we allow the function badly behaved nearby boundary even if we impose continuity (since this is local information). But if the function is continuous on a closed and bounded interval, the structure of domain limits the possibility of bad behavior. The second function is bounded. And this is true in general.

Theorem 1.1. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, then there is $M > 0$ such that $|f(x)| \le M$ for all $x \in [a, b]$.*

Remark 1.1. One might compare the local boundedness theorem in previous lecture: If f is continuous at $c \in A$, then there is $\delta_c, M_c > 0$ such that $|f(x)| \le M_c$ for all $x \in A, |x - c| < \delta_c$. This local boundedness theorem doesn't imply the global boundedness as can be seen from the example $f(x) = x^{-1}$. This is because the δ_c found using continuity depends on the center c. As $c \to \partial A$, δ_c might degenerate, and M_c might blow up to $+\infty$ which gives us no information. (Think about the explicit value of δ_c in the example $f(x) = x^{-1}$ on $(0, 1]$).

Remark 1.2. As mentioned above, the assumption of closeness is necessary, $f(x) = x^{-1}$ on $(0, 1]$ is unbounded but is continuous. The boundedness is also necessary, as can be seen from $f(x) = x$ on $[0, +\infty)$.

Thanks to the boundedness Theorem, it is clear from completeness axiom that both

$$M = \sup\{f(x) : x \in [a,b]\}, \quad m = \inf\{f(x) : x \in [a,b]\} \tag{1.1}$$

exists as a real number. The next Theorem shows that m, M can in fact be achieved.

Theorem 1.2 (Max-Min Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function, then there is $x_1, x_2 \in [a,b]$ such that $f(x_1) = M$ and $f(x_2) = m$ so that for all $x \in [a,b]$,*

$$f(x_1) \le f(x) \le f(x_2).$$

Sketch of Proof. (Refer to Textbook if you wish more detail) By definition of sup, take x_i such that $f(x_i) \to M$. Since $x_i \in [a,b]$, $x_{i_k} \to \bar{x}$ for some $\bar{x} \in [a,b]$. By Sequence criterion, $f(x_{i_k}) \to f(\bar{x}) = M$. The lower bound is similar. □

Some variation of Max-Min Theorem: Given a continuous function $f : [a,b] \to \mathbb{R}$. How can we find $\bar{x}$ such that $f(\bar{x}) = 0$? First of all, if $f(x) > 0$ or < 0 for all x, this is clearly impossible. If $f \equiv 0$, then the assertion is trivial. What if f is positive and negative somewhere?

Theorem 1.3. *Suppose $f : [a,b] \to \mathbb{R}$ is a continuous such that $f(a) > k > f(b)$ for some $k \in \mathbb{R}$, then there is $\bar{x} \in [a,b]$ such that $f(\bar{x}) = k$.*

Proof. By translation, we may assume $k = 0$. In the textbook (or in class), we use the bisection method which is a algorithm to locate the root. Here I am going to give an alternative proof (also discussed in class).

Let $p = \sup S = \sup\{s \in [a,b] : f(x) > 0 \quad \forall x \in [a,s]\}$. Since $a \in S$, $p \in \mathbb{R}$ exists by completeness. It suffices to show that $f(p) = 0$, namely is the first root. By continuity at a, there exists $\delta_a > 0$ such that $f(x) > 0$ on $[a, a + \delta_a)$ so that $p > a$. Like-wisely, $p < b$ by continuity at b.

Assume $f(p) > 0$, then there is $\delta > 0$ such that for all $x \in (p-\delta, p+\delta) \subset [a,b]$, we have $f(x) > 0$. Since $p - \delta < p = \sup S$, there is $s_0 \in S$ such that $p - \delta < s_0$ and hence we have $f(x) > 0$ on $[a, p + \delta)$. This implies $p + \delta/2 \in S$ which is impossible.

Assume $f(p) < 0$, then similarly there is $\delta > 0$ such that for all $x \in (p-\delta, p+\delta) \subset [a,b]$, we have $f(x) < 0$. By the same argument, we have $f(x) > 0$ on $[a, p - \delta/2]$ which is impossible. Therefore, we must have $f(p) = 0$! □