Math 2050, HW 4, Due: 6 Nov

(1) Let $x_1 < x_2$ be two given real numbers. Define the sequence inductively by

$$x_n = \frac{1}{3}x_{n-1} + \frac{2}{3}x_{n-2}$$

for all $n > 2$, show that $\{x_n\}$ is convergent and find the limit.

(2) If $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{x_n}$ for all $n \geq 1$, show that $\{x_n\}$ is a con tractive sequence, i.e. there exists $C \in [0, 1)$ such that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|.$$

Show that $\{x_n\}$ is convergent and find the limit.

(3) Find an example of sequence $\{x_n\}$ such that it is not a Cauchy sequence but for any fixed $p \in \mathbb{N}$, $x_{n+p} - x_n \to 0$ as $n \to +\infty$.

(4) Show that if $x_n > 0$ for all $n \in \mathbb{N}$, then $x_n \to 0$ as $n \to +\infty$ if and only if $x_n^{-1} \to +\infty$ as $n \to +\infty$.