8/11/23

MATH2050A Tutorial

Announcements:

- Midterm 2: 15/11. Tutorial next week will be on Friday 17/11.

Midterm Review

Defn: $\{x_n\}$ converges to $x \in \mathbb{R}$ if $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ s.t. if $n \geq N$, $|x_n - x| < \varepsilon$.

- $\{x_n\}$ is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
  decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$
  monotone if it is increasing or decreasing.
- let $\{x_n\}$ be a sequence and $n_1 < n_2 < \cdots < n_k < \cdots$ be a sequence of natural numbers then the sequence given by $(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$ is a subsequence of $\{x_n\}$.

$\{x_{n_k}\}$.

$\{x_n\}$: $f: \mathbb{N} \to \mathbb{R}$. $f(1) = x_1$
$f(2) = x_2$
$\vdots$

Subsequence is $f \circ g: \mathbb{N} \to \mathbb{R}$
where $g: \mathbb{N} \to \mathbb{N}$ and is a strictly increasing
fn. $g(1) = n_1$, $g(2) = n_2$, ...

$$f \circ g(1) = f(n_1) = x_{n_1}$$
$$f \circ g(2) = f(n_2) = x_{n_2}$$
$$\vdots$$
$$f \circ g(k) = f(n_k) = x_{n_k}.$$

- Given a sequence $\{x_n\}$. $\limsup_{n\to\infty}(x_n) = \inf_{n\geq 0} \sup_{k\geq n} x_k$
- $\{x_n\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ s.t. if $m, n \geq N$, $|x_m - x_n| < \varepsilon$.

Thms:

- Bolzano-Weierstrass: Any bounded sequence has convergent subsequence.
- Monotone Convergence Thm: A monotone sequence converges iff it is bounded.
- Cauchy Criterion: A sequence converges iff it is Cauchy.
- If $\{x_n\}$ converges to x, then all subsequences converge to x.
- $\{x_n\}$ does not converge/is divergent if either

- $\{x_n\}$ is unbounded
- $\{x_n\}$ has two subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ where $\lim_{k\to\infty} x_{n_k} \neq \lim_{k\to\infty} x_{m_k}$.

• Equivalent def'ns of limsup.

• Squeeze Thm.

Q1: Show that if $\{x_n\}$ is unbounded, then $\exists$ subsequence $\{x_{n_k}\}$ s.t. $\lim_{k\to\infty} \frac{1}{x_{n_k}} = 0$.

Pf: $\{x_n\}$ is unbounded, so $\forall M \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $|x_n| > M$.

So for each $K \in \mathbb{N}$, choose n_k s.t. $|x_{n_k}| > K$, $n_k > n_{k-1}$.

So consider the sequence $\left\{\frac{1}{x_{n_k}}\right\}$. WTS $\lim_{k\to\infty} \frac{1}{x_{n_k}} = 0$.

Let $\varepsilon > 0$. By AP, $\exists K \in \mathbb{N}$ s.t. $\frac{1}{K} < \varepsilon$. So $\exists n_k$ s.t. if $n_m \geq n_k$,

$$\left|\frac{1}{x_{n_m}}\right| \leq \left|\frac{1}{x_{n_k}}\right| \leq \frac{1}{K} < \varepsilon.$$

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Q2: Show that $x_n = \sqrt{n}$ satisfies $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ but it is not Cauchy.

Pf: $\{x_n\}$ is not Cauchy because it is not bounded and therefore diverges.

Let $M \in \mathbb{R}$. Then by A.P. of $\mathbb{N}$, $\exists N \geq M^2$. So $|x_N| = |\sqrt{N}| \geq M$.

$|x_{n+1} - x_n| = |\sqrt{n+1} - \sqrt{n}| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| \longrightarrow 0$ as $n \to \infty$ by A.P.

Observe: $\sqrt{n+1} - \sqrt{n} = \frac{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} - \frac{\sqrt{n}(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$