1/11/23

MATH2050A Tutorial

Announcements:

- HW4 due 6/11
- Midterm 2: 15/11

Recall Def: $\{x_n\}_{n=1}^{\infty}$ is Cauchy if $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $\forall\, m,n\geq N$, $|x_m-x_n|<\varepsilon$.

Thm: Cauchy $\Longleftrightarrow$ convergence. (Cauchy is also a way we could have defined completeness property of $\mathbb{R}$).

Q1: Show directly from the definition that $x_n=\frac{n+1}{n}$ is Cauchy.

Pf: WLOG we can take $m>n$. Since $m>n$, $|n-m|\leq|m|$.

$$\left|x_m-x_n\right| = \left|\frac{m+1}{m}-\frac{n+1}{n}\right| = \left|\frac{mn+n-mn-m}{mn}\right| = \left|\frac{n-m}{nm}\right| \leq \left|\frac{m}{nm}\right| = \left|\frac{1}{n}\right|.$$

Let $\varepsilon>0$ be given. Then choosing $N>\frac{1}{\varepsilon}$, $\forall\, m,n\geq N$,

$$|x_m-x_n| \leq \left|\frac{1}{n}\right| < \varepsilon.$$

$\checkmark$

Q2: Show that if $\{x_n\}$, $\{y_n\}$ are Cauchy sequences, then $\{x_n+y_n\}$ is Cauchy.

Pf: Let $\varepsilon>0$ be given. Then since $\{x_n\}$, $\{y_n\}$ are Cauchy, there are $N_x, N_y \in \mathbb{N}$ s.t.

$\forall m,n \geq N_x$, $|x_m - x_n| < \frac{\varepsilon}{2}$,

$m,n \geq N_y$, $|y_m - y_n| < \frac{\varepsilon}{2}$.

So then for $m,n \geq \max\{N_x, N_y\}$

$$|x_m+y_m-x_n-y_n| \leq |x_m-x_n| + |y_m-y_n| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Def: $\{x_n\}$ is contractive if $\exists$ a constant C, $0<C<1$, s.t.

$$|x_{n+2}-x_{n+1}| \leq C|x_{n+1}-x_n| \quad \text{for all } n \in \mathbb{N}.$$

Q3: Prove that every contractive sequence is Cauchy. (and hence converges).

Pf: Let $\varepsilon>0$ be given. for any $m>n$,

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \cdots + x_{n+1} - x_n|$$

$$\overset{\Delta}{\leq} |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|.$$

In general, $|x_{n+2} - x_n| \leq C|x_{n+1} - x_n| \leq C^2 |x_n - x_{n-1}| \leq \cdots \leq C^n |x_2 - x_1|$

$$\leq C^{m-2}|x_2 - x_1| + C^{m-3}|x_2 - x_1| + \cdots + C^{n-1}|x_2 - x_1|$$

$$= (C^{m-2} + C^{m-3} + \cdots + C^{n-1})|x_2 - x_1|$$

$$= C^{n-1}(1 + C + C^2 + \cdots + C^{m-n-1})|x_2 - x_1|$$

$$= C^{n-1}\left(\frac{1 - C^{m-n}}{1-C}\right)|x_2 - x_1| \underset{C<1}{\leq} \underbrace{C^{n-1}\left(\frac{1}{1-C}\right)|x_2 - x_1|}_{\leq M.}$$

$$\leq C^{n-1} M.$$

Since $0<C<1$, $\lim_{n\to\infty} C^{n-1} = 0$, so $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$, $C^{n-1} < \frac{\varepsilon(1-C)}{|x_2-x_1|}$

So for $m, n \geq N$,

$$|x_m - x_n| \leq C^{n-1}\left(\frac{1}{1-C}\right)|x_2-x_1| < \frac{\varepsilon(1-C)}{|x_2-x_1|} \frac{|x_2-x_1|}{1-C} = \varepsilon.$$