§ 6.2

6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin \mathsf{W}$, prove that there exists $y \in \mathsf{V}$ such that $y \in \mathsf{W}^\perp$, but $\langle x, y\rangle \neq 0$. *Hint:* Use Theorem 6.6.

13. Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Prove the following results.

(a) $S_0 \subseteq S$ implies that $S^\perp \subseteq S_0^\perp$.
(b) $S \subseteq (S^\perp)^\perp$; so $\text{span}(S) \subseteq (S^\perp)^\perp$.
(c) $\mathsf{W} = (\mathsf{W}^\perp)^\perp$. *Hint:* Use Exercise 6.
(d) $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}^\perp$. (See the exercises of Section 1.3.)

14. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(\mathsf{W}_1+\mathsf{W}_2)^\perp = \mathsf{W}_1^\perp \cap \mathsf{W}_2^\perp$ and $(\mathsf{W}_1 \cap \mathsf{W}_2)^\perp = \mathsf{W}_1^\perp + \mathsf{W}_2^\perp$. (See the definition of the sum of subsets of a vector space on page 22.) *Hint for the second equation:* Apply Exercise 13(c) to the first equation.

Assume $(W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$

Since $W_1^\perp, W_2^\perp$ are finite-dim. $(W_1^\perp + W_2^\perp)^\perp = (W_1^\perp)^\perp \cap (W_2^\perp)^\perp$

By 13(c). $W_1 = (W_1^\perp)^\perp$ $\quad W_2 = (W_2^\perp)^\perp$ $\quad W_1^\perp + W_2^\perp = ((W_1^\perp + W_2^\perp)^\perp)^\perp$

Therefore. $W_1^\perp + W_2^\perp = ((W_1^\perp + W_2^\perp)^\perp)^\perp = ((W_1^\perp)^\perp \cap (W_2^\perp)^\perp)^\perp$

$= (W_1 \cap W_2)^\perp$

23. Let V be the vector space defined in Example 5 of Section 1.2, the space of all sequences σ in F (where $F = R$ or $F = C$) such that $\sigma(n) \neq 0$ for only finitely many positive integers n. For $\sigma, \mu \in \mathsf{V}$, we define $\langle \sigma, \mu\rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}$. Since all but a finite number of terms of the series are zero, the series converges.

(a) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V, and hence V is an inner product space.
(b) For each positive integer n, let e_n be the sequence defined by $e_n(k) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta. Prove that $\{e_1, e_2, \ldots\}$ is an orthonormal basis for V.
(c) Let $\sigma_n = e_1 + e_n$ and $\mathsf{W} = \text{span}(\{\sigma_n : n \geq 2\}$.
(i) Prove that $e_1 \notin \mathsf{W}$, so $\mathsf{W} \neq \mathsf{V}$.
(ii) Prove that $\mathsf{W}^\perp = \{0\}$, and conclude that $\mathsf{W} \neq (\mathsf{W}^\perp)^\perp$.

(a) easy to check with the def of inner product.

(b) $\langle e_i, e_j \rangle = \sum_{n=1}^{\infty} e_i(n) \cdot \overline{e_j(n)} = \sum_{n=1}^{\infty} \delta_{i,n} \cdot \bar{\delta}_{j,n} = \delta_{ij}$

So $\{e_i\}_1^{\infty}$ is orthonormal subset of V

$\forall\, \sigma \in V$. $S = \{ n : \sigma(n) \neq 0 \}$ is a finite set

Then $\sigma = \sum_{n \in S} \sigma(n) \cdot e_n \in \text{span}(\{e_i\}_1^{\infty})$

Therefore. $\{e_i\}_1^{\infty}$ is an orthonormal basis for V.

(c)

(i) if $e_1 \in W$, then $e_1 = a_2 \cdot \sigma_2 + \cdots + a_n \cdot \sigma_n$ for some $a_2 \cdots a_n \in F$

Then $(a_2 + \cdots + a_n - 1) e_1 + a_2 e_2 + \cdots + a_n e_n = 0$

Since $\{e_1 \cdots e_n\} \subset \{e_j\}_1^{\infty}$ is L.I.

we have $a_2 + \cdots + a_n - 1 = a_2 = \cdots = a_n = 0$ which is impossible.

So $e_1 \notin W$ and thus $W \neq V$

(ii)

$\forall\, v \in W^{\perp}$. we have $\langle v, \sigma_n \rangle = 0$ for $n \geq 2$

i.e. $\langle v, e_1 \rangle = - \langle v, e_n \rangle \quad \forall n \geq 2$

i.e. $v(1) = -v(n) \quad \forall n \geq 2$

If $v(1) \neq 0$, Then $v(n) \neq 0 \quad \forall n \geq 2$.

which implies $v \notin V$. Contradiction!

Thus $v(1) = 0$ and $v(n) = -v(1) = 0 \quad \forall n \geq 2$

i.e. $v = 0 \in V$.

Thus $W^{\perp} = \{0\}$

we conclude that $W \neq V = \{0\}^{\perp} = (W^{\perp})^{\perp}$

Therefore, that W is finite-dim is essential in exercise 13(c)

§ 6.3

9. Prove that if $V = W \oplus W^\perp$ and T is the projection on W along $W^\perp$, then $T = T^*$. *Hint:* Recall that $N(T) = W^\perp$. (For definitions, see the exercises of Sections 1.3 and 2.1.)

$\forall\, v \in V.\quad \exists!\, x_1 \in W,\ y_1 \in W^\perp$ s.t $\quad T(v) = x_1$

$\forall\, w \in V.\quad \exists!\, x_2 \in W,\ y_2 \in W^\perp$ s.t $\quad T(w) = x_2$

$$\begin{cases} \langle T(v), w \rangle = \langle x_1, x_2 + y_2 \rangle = \langle x_1, x_2 \rangle \\ \langle T^*(v), w \rangle = \langle v, T(w) \rangle = \langle x_1 + y_1, x_2 \rangle = \langle x_1, x_2 \rangle \end{cases}$$

So $\langle T(v), w \rangle = \langle T^*(v), w \rangle \quad \forall\, v, w \in V.$

So $T = T^*$

12. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.

(a) $\mathsf{R}(\mathsf{T}^*)^\perp = \mathsf{N}(\mathsf{T})$.

(b) If V is finite-dimensional, then $\mathsf{R}(\mathsf{T}^*) = \mathsf{N}(\mathsf{T})^\perp$. *Hint:* Use Exercise 13(c) of Section 6.2. $R(T^*) = (R(T^*)^\perp)^\perp = N(T)^\perp$

Remark : $U^\perp = W \not\Leftrightarrow U = W^\perp$ for general $U, W \subset V$

13. Let T be a linear operator on a finite-dimensional vector space V. Prove the following results.

(a) $\mathsf{N}(\mathsf{T}^*\mathsf{T}) = \mathsf{N}(\mathsf{T})$. Deduce that $\text{rank}(\mathsf{T}^*\mathsf{T}) = \text{rank}(\mathsf{T})$.

(b) $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{T}^*)$. Deduce from (a) that $\text{rank}(\mathsf{T}\mathsf{T}^*) = \text{rank}(\mathsf{T})$.

(c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

(a) $\forall v \in N(T)$. $T^*T(v) = T^*(T(v)) = T^*(0) = 0$

So $v \in N(T^*T)$ $\quad N(T) \subset N(T^*T)$

$\forall v \in N(T^*T)$ $\quad \|T(v)\|^2 = \langle T(v), T(v) \rangle = \langle T^*T(v), v \rangle = \langle 0, v \rangle = 0$

i.e. $T(v) = 0$, $\quad v \in N(T)$. $\quad N(T^*T) \subset N(T)$

$\text{rank}(T^*T) = \dim(V) - \text{nullity}(T^*T) = \dim(V) - \text{nullity}(T)$
$= \text{rank}(T)$

(b) By Q12(b), $R(T^*) = N(T)^\perp$

Besides. $V = N(T) \oplus N(T)^\perp$ So $\dim(V) = \dim(N(T)) + \dim(N(T)^\perp)$

$\text{rank}(T) = \dim(V) - \text{nullity}(T) = \dim(N(T)^\perp) = \dim(R(T^*)) = \text{rank}(T^*)$

$\text{rank}(TT^*) = \text{rank}((TT^*)^*) = \text{rank}(T^*T) = \text{rank}(T)$

§ 6.4.

10. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in \mathsf{V}$

$$\|\mathsf{T}(x) \pm ix\|^2 = \|\mathsf{T}(x)\|^2 + \|x\|^2.$$

Deduce that $\mathsf{T} - i\mathsf{I}$ is invertible and that $[(\mathsf{T} - i\mathsf{I})^{-1}]^* = (\mathsf{T} + i\mathsf{I})^{-1}$.

- T is self-adjoint. then $T = T^*$ and $\langle T(x), x \rangle = \langle x, T(x) \rangle$
- $\| T(x) \pm ix \|^2 = \langle T(x) \pm ix, \ T(x) \pm ix \rangle$

$$= \| T(x) \|^2 + \langle T(x), \pm ix \rangle + \langle \pm ix, T(x) \rangle + \| \pm ix \|^2$$

$$= \| T(x) \|^2 \mp i \langle T(x), x \rangle \pm i \langle x, T(x) \rangle + \| x \|^2$$

$$= \| T(x) \|^2 + \| x \|^2$$

- $\forall x \in N(T - iI)$

$$0 \le \| x \|^2 \le \| T(x) \|^2 + \| x \|^2 = \| T(x) - ix \|^2 = \| (T - iI)(x) \|^2 = 0$$

So $x = 0$ i.e. $T - iI$ is 1-1 and thus invertible.

- Since $T = T^*$, $(T - iI)^* = T^* + iI^* = T + iI$

- $\langle x, [(T - iI)^{-1}]^* (T + iI)(y) \rangle$

$$= \langle (T - iI)^{-1}(x), (T + iI)(y) \rangle$$

$$= \langle (T - iI)^{-1}(x), (T - iI)^*(y) \rangle$$

$$= \langle (T - iI)(T - iI)^{-1}(x), y \rangle$$

$$= \langle x, y \rangle \quad \text{for any } x, y \in V$$

Thus $[(T - iI)^{-1}]^* \circ (T + iI) = I$

$$[(T - iI)^{-1}]^* = (T + iI)^{-1}$$