$T: V \to V$. β is an ordered basis for V.

λ is a eig. val of T.

$$\begin{cases} E_\lambda = N(T - \lambda I) \overset{\text{sub}}{\subset} V \\ G_\lambda = N([T]_\beta - \lambda I_n) \overset{\text{sub}}{\subset} \mathbb{R}^n \end{cases}$$

Show that $\Psi: E_\lambda \to G_\lambda$ is isomorphism.
$v \mapsto [v]_\beta$

$$\boxed{\begin{array}{l} \text{find } E_\lambda \\ \Leftrightarrow \text{find basis for } E_\lambda \\ \Leftrightarrow \text{find basis for } G_\lambda \end{array}}$$

Proof.

- well-defined.

  $\forall v \in E_\lambda$. $(T - \lambda I)v = \vec{0}$

  $[T]_\beta [v]_\beta = \lambda \cdot [v]_\beta$.

  $\Psi(v) = [v]_\beta \in N([T]_\beta - \lambda I_n) = G_\lambda$.

- Ψ is linear

- $\forall \vec{a} \in G_\lambda$. $[T]_\beta \vec{a} = \lambda \vec{a}$

  let $v = a_1 v_1 + \cdots + a_n v_n$, then $[v]_\beta = \vec{a}$

  $[T(v)]_\beta = [T]_\beta [v]_\beta = [T]_\beta \vec{a} = \lambda \vec{a} = [\lambda v]_\beta$

  $T(v) = \lambda v$. $v \in N(T - \lambda I) = E_\lambda$

  So Ψ is onto.

- $v \in N(\Psi)$. $\Psi(v) = \vec{0}$, $[v]_\beta = \vec{0}$

  $v = 0 v_1 + \cdots + 0 v_n = 0_V$ $\therefore$ Ψ is 1-1

1. Let T be a diagonalizable linear operator on a finite-dimensional vector space V over F, and let $f, g \in P(F)$. Prove that $f(T)$ and $g(T)$ are simultaneously diagonalizable.

Def: $T: V \to V$, $U: V \to V$ be linear.

T and U are simultaneously Diagonalizable

if $\exists$ basis β for V s.t. $[T]_\beta$ and $[U]_\beta$ are diagonal.

Proof. T is diagonalizable.

$\exists\ \beta$, eigen basis of T. st $[T]_\beta = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$[f(T)]_\beta = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} \qquad [g(T)]_\beta = \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & g(\lambda_n) \end{bmatrix}$$

Furthurmore.

$$[f(T) \circ g(T)]_\beta = \begin{bmatrix} f(\lambda_1) \cdot g(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \cdot g(\lambda_n) \end{bmatrix} = \begin{bmatrix} g(\lambda_1) f(\lambda_1) & & \\ & \ddots & \\ & & g(\lambda_n) \cdot f(\lambda_n) \end{bmatrix}$$

$$= [g(T) \circ f(T)]_\beta$$

So $f(T) \circ g(T) = g(T) \circ f(T)$

20. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then UT = TU if and only if U = g(T) for some polynomial $g(t)$. *Hint:* Suppose that V is generated by v. Choose $g(t)$ according to Exercise 13 so that $g(\mathsf{T})(v) = \mathsf{U}(v)$.

V is T-cylic subspace of itself

$\Leftrightarrow \exists\, v_0 \in V$. st $V = \text{span}(\{v_0, T(v_0), T^2(v_0), \ldots\})$

Pf. $UT = TU$

Since $U : V \to V$, $U(v_0) \in V = \text{span}(\{v_0, T(v_0), \ldots\})$

So $U(v_0) = a_0 v_0 + a_1 T(v_0) + \cdots + a_{k-1} T^{k-1}(v_0)$ for some $a_0 \cdots a_{k-1}$

Let $g(t) = a_0 + a_1 t + \cdots + a_{k-1} t^{k-1}$

We show that $U = g(T)$

which is equivalent to show $U(T^j(v_0)) = g(T)(T^j(v_0))$ $\forall j \geq 0$

$j=0$. $U(v_0) = g(T)(v_0)$

$j>0$
$$\begin{aligned} U(T^j(v_0)) &= U \circ T \circ T^{j-1}(v_0) \\ &= T \circ U \circ T^{j-1}(v_0) \\ &\vdots \\ &= T^j U(v_0) \\ &= T^j \circ g(T)(v_0) \\ &= g(T) \circ T^j(v_0) \\ &= g(T)(T^j(v_0)) \end{aligned}$$

U agrees with $g(T)$ on $\{v_0, T(v_0), \ldots\}$

So $U = g(T)$

21. Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or T = cI for some scalar c.

If V is not a T-cyclic subspace of itself.

Then $\forall$ nonzero $v \in V$. $\underbrace{\mathrm{span}\{v, T(v), \cdots\}}_{\dim < 2} \subsetneq \underbrace{V}_{\dim = 2}$.

$v \neq 0$ so $\dim(\mathrm{span}\{v, T(v), \cdots\}) = 1$ $T(v) = \lambda_v \cdot v$.

Claim: λ_v is independent of v. i.e. $\lambda_v = \lambda$

Choose $\{v_1, v_2\}$ basis for V. Then $v_1 + v_2 \neq 0$

$$\begin{cases} T(v_1) = \lambda_1 v_1 \\ T(v_2) = \lambda_2 v_2 \\ T(v_1 + v_2) = \lambda_3 (v_1 + v_2) \end{cases}$$

So $\lambda_1 v_1 + \lambda_2 v_2 = \lambda_3 (v_1 + v_2)$

$(\lambda_1 - \lambda_3) v_1 + (\lambda_2 - \lambda_3) v_2 = 0$

Since $\{v_1, v_2\}$ L.I. , $\lambda_1 - \lambda_3 = \lambda_2 - \lambda_3 = 0$

$\lambda_1 = \lambda_2 = \lambda_3 =: \lambda$

$$\begin{cases} T(v_1) = \lambda v_1 \\ T(v_2) = \lambda v_2 \end{cases}$$ So $T = \lambda I$.

26. Let T be a linear operator on an n-dimensional vector space V such that T has n distinct eigenvalues. Prove that V is a T-cyclic subspace of itself. *Hint:* Use Exercise 23 to find a vector v such that $\{v, \mathsf{T}(v), \ldots, \mathsf{T}^{n-1}(v)\}$ is linearly independent.

$\lambda_1 \cdots \lambda_n \qquad T(v_i) = \lambda_i v_i. \qquad \lambda_i \neq \lambda_j \text{ if } i \neq j.$

Let $v = v_1 + v_2 + \cdots + v_n.$

$$T^i(v) = T^i(v_1) + \cdots + T^i(v_n) = \lambda_1^i v_1 + \cdots + \lambda_n^i v_n$$

Consider

$$a_0 v + a_1 T(v) + \cdots + a_{n-1} T^{n-1}(v) = \vec{0}$$

$$a_0(v_1 + \cdots + v_n) + \cdots + a_{n-1}(\lambda_1^{n-1} v_1 + \cdots + \lambda_n^{n-1} v_n) = \vec{0}$$

$$(a_0 + a_1\lambda_1 + \cdots + a_{n-1}\lambda_1^{n-1}) v_1 + \cdots + (a_0 + a_1\lambda_n + \cdots + a_{n-1}\lambda_n^{n-1}) v_n = \vec{0}$$

$\{v_1 \cdots v_n\}$ L.I.

so
$$\begin{cases} a_0 + a_1\lambda_1 + \cdots + a_{n-1}\lambda_1^{n-1} = 0 \\ \vdots \\ a_0 + a_1\lambda_n + \cdots + a_{n-1}\lambda_n^{n-1} = 0 \end{cases}$$

i.e.
$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = \vec{0} \Rightarrow a_0 = \cdots = a_{n-1} = 0$$

so $\{v, T(v), \ldots, T^{n-1}(v)\}$ L.I. is basis