Sec 2.5 Q13

13.† Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \quad \text{for } 1 \le j \le n,$$

and set $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

Given basis β

- For any basis β', $[I_V]_{\beta'}^{\beta}$ change of coord matrix is invertible
- For any invertible Q, $\exists\, \beta'$ basis for V. s.t. $[I]_{\beta'}^{\beta} = Q$?

1-1 correspondence between basis β' and invertible Q.

- β' is L.I.

Consider $a_1 \cdot x_1' + \cdots + a_n \cdot x_n' = 0$

$$0 = \sum_{j=1}^{n} a_j \cdot x_j' = \sum_j a_j \cdot \sum_i Q_{ij} x_i = \sum_i \Big(\sum_j Q_{ij} a_j\Big) x_i$$

Since β is basis. L.I, $\sum_j Q_{ij} a_j = 0 \quad i = 1, \ldots, n$

i.e $Q \cdot \vec{a} = \vec{0} \qquad \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

Since Q invertible, $\vec{a} = \vec{0}$ $\therefore a_1 = \cdots = a_n = 0$. β' L.I

- $\beta' \subset V$. $\dim(V) = n = |\beta'|$ Thus β' is basis

- $$\left([I_V]_{\beta'}^{\beta}\right)_{ij} = \left([I_V(x_j')]_{\beta}\right)_i = \begin{pmatrix} Q_{1j} \\ \vdots \\ Q_{nj} \end{pmatrix}_i = Q_{ij}$$

1. Sec. 2.2 Q15

15. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{\mathsf{T} \in \mathcal{L}(\mathsf{V},\mathsf{W})\colon \mathsf{T}(x) = 0 \text{ for all } x \in S\}$. Prove the following statements.

(a) S^0 is a subspace of $\mathcal{L}(\mathsf{V},\mathsf{W})$.
(b) If $S_1$and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.
(c) If V_1 and V_2 are subspaces of V, then $(\mathsf{V}_1+\mathsf{V}_2)^0 = \mathsf{V}_1^0 \cap \mathsf{V}_2^0$.

(a) • Let T_0 be the zero transformation from V to W.
Then $T_0(x)=0 \quad \forall x \in S \subset V$.
So $T_0 \in S^0$

• $\forall\ T_1, T_2 \in S^0. \quad \forall c \in F$
$(cT_1+T_2)(x) = c\cdot T_1(x) + T_2(x) = c\cdot 0+0=0 \quad \forall x \in S$
So $cT_1+T_2 \in S^0$

Thus S^0 is a subspace of $\mathcal{L}(V,W)$

(b) $\forall\ T \in S_2^0. \quad T(x)=0 \quad \forall x \in S_2$
Since $S_1 \subset S_2$, $\quad T(x)=0 \quad \forall x \in S_1 \subset S_2$
Thus $T \in S_1^0$. i.e. $S_2^0 \subset S_1^0$

(c) • Since $V_1 \subset V_1+V_2$, $V_2 \subset V_1+V_2$
By (b) we have $(V_1+V_2)^0 \subset V_1^0$ and $(V_1+V_2)^0 \subset V_2^0$
Then $(V_1+V_2)^0 \subset V_1^0 \cap V_2^0$

• $\forall\ T \in V_1^0 \cap V_2^0$ Then $T(x)=0 \quad \forall x \in V_1 \cup V_2$
Then for any $y \in V_1+V_2$, there exist $x_1 \in V_1$, $x_2 \in V_2$
such that $y = x_1+x_2$
Then $T(y) = T(x_1)+T(x_2) = 0+0=0$
Thus $T \in (V_1+V_2)^0$ i.e. $V_1^0 \cap V_2^0 \subset (V_1+V_2)^0$

Sec. 2.6 Q20

20. Let V and W be nonzero vector spaces over the same field, and let $T: V \to W$ be a linear transformation.

(a) Prove that T is onto if and only if T^t is one-to-one.
(b) Prove that T^t is onto if and only if T is one-to-one.

Hint: Parts of the proof require the result of Exercise 19 for the infinite-dimensional case.

$T: V \to W$ $\quad$ $T^*: W^* \to V^*$

(a) ($\Rightarrow$) suppose T is onto

$\forall w \in W$, $\exists v \in V$ s.t. $T(v) = w$

$\forall g \in N(T^*)$ $\quad gT = T^*(g) = 0$ $\quad$ i.e. $\forall v \in V$, $g(T(v)) = 0$

since T onto. $\forall w \in W$ $\quad g(w) = g(T(v)) = 0$ Hence $g = 0$

That means $N(T^*) = \{0\}$, so T^* is 1-1

($\Leftarrow$) if T is not onto. then $R(T) \overset{\text{proper}}{\subset} W$

$\exists$ non zero $g_0 \in W^*$ st $g_0(w) = 0$ $\forall w \in R(T)$ $\quad$ i.e. $g_0 T = 0$

$\exists w' \in W \setminus R(T)$ s.t. $g_0(w') \neq 0$

$T^*(g) = gT = gT + g_0 T = (g+g_0)T = T^*(g+g_0)$

but $g(w') \neq (g+g_0)(w')$ so $g \neq g+g_0$

Hence T^* is not 1-1.

(b) ($\Rightarrow$) suppose T^* is onto.

$\forall f \in V^*$ $\exists g \in W^*$ s.t. $T^*(g) = f \Rightarrow gT = f$

$\forall x \in N(T)$ we have $f(x) = gT(x) = g(0) = 0$, $\forall f \in V^*$

Then $x = 0$. that means $N(T) = \{0\}$. Hence T is 1-1

$(\Leftarrow)$ Let β be a basis for V.

T is 1-1. so $T(\beta)$ is L.I.

Extend $T(\beta)$ to $T(\beta) \cup \gamma$ a basis for W.

$$T^*: W^* \to V^*$$
$$g \mapsto g \circ T$$

$\forall f \in V^*$, define $g \in W^*$

$$g(w) = \begin{cases} f \circ T^{-1}(w) & w \in T(\beta) \\ 0 & w \in \gamma \end{cases}$$

Then

$$T^*(g)(v) = g \circ T(v) \overset{w = T(v) \in T(\beta)}{=} f \circ T^{-1} \circ T(v) = f(v) \quad \forall v \in \beta$$

so $T^*(g) = f$. T^* is onto.