Sec. 2.3: Q17

17. Let V be a vector space. Determine all linear transformations $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T} = \mathsf{T}^2$. *Hint:* Note that $x = \mathsf{T}(x) + (x - \mathsf{T}(x))$ for every x in V, and show that $\mathsf{V} = \{y\colon \mathsf{T}(y) = y\} \oplus \mathsf{N}(\mathsf{T})$ (see the exercises of Section 1.3).

Definition. *Let* V *be a vector space and* W_1 *and* W_2 *be subspaces of* V *such that* $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$. *(Recall the definition of direct sum given in the exercises of Section 1.3.) A function* $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ *is called the* **projection on** W_1 **along** W_2 *if, for* $x = x_1 + x_2$ *with* $x_1 \in \mathsf{W}_1$ *and* $x_2 \in \mathsf{W}_2$*, we have* $\mathsf{T}(x) = x_1$.

Proof. Claim: $T = T^2 \iff T$ is a projection

($\Rightarrow$) suppose $T = T^2$. Let $W_T = \{y \in V : T(y) = y\}$

we will show that $V = W_T \oplus N(T)$

① $\forall x \in W_T \cap N(T)$.

$$\begin{cases} x \in W_T, & \text{so } x = T(x). \\ x \in N_T, & \text{so } T(x) = 0 \end{cases} \Rightarrow x = 0$$

so $W_T \cap N(T) = \{0\}$

② $\forall x \in W_T$, $x = T(x) \in R(T)$, so $W_T \subset R(T)$

$\forall x \in V$, $T(T(x)) = T^2(x) = T(x)$ so $T(x) \in W_T$ i.e. $R(T) \subset W_T$

Therefore $W_T = R(T)$

Besides, $T(x - T(x)) = T(x) - T^2(x) = 0 \quad \forall x \in V$

so $x - T(x) \in N(T) \quad \forall x \in V$.

Since $x = \underbrace{T(x)}_{\in W_T} + \underbrace{(x - T(x))}_{\in N(T)} \quad \forall x \in V$

then $V = W_T + N(T)$

By ① and ②, $V = W_T \oplus N(T) = R(T) \oplus N(T)$

T is projection on W_T along $N(T)$

($\Leftarrow$) suppose T is a projection on W_1 along W_2.

then $V = W_1 \oplus W_2$

For any $v \in V$. $\exists!\ w_1 \in W_1$, $w_2 \in W_2$

s.t. $v = w_1 + w_2$, and $T(v) = w_1$

Since $w_1 = \underset{\in W_1}{w_1} + \underset{\in W_2}{0}$, we have $T(w_1) = w_1$

So $T^2(v) = T(T(v)) = T(w_1) = w_1 = T(v)$

i.e $T^2 = T$

5. Sec. 2.4: Q16

16. Let B be an $n \times n$ invertible matrix. Define $\Phi\colon \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Proof:

- Φ is linear.

$\forall A_1, A_2 \in M_{n\times n}(F)$, $\forall a \in F$

$$\begin{aligned}\Phi(a\cdot A_1 + A_2) &= B^{-1}(aA_1 + A_2)B \\ &= B^{-1}\cdot(aA_1B + A_2B) \\ &= a\cdot B^{-1}A_1B + B^{-1}A_2B \\ &= a\cdot \Phi(A_1) + \Phi(A_2)\end{aligned}$$

- Φ is injective

$\forall A \in N(\Phi)$. $\Phi(A) = O_{n\times n}$ i.e. $B^{-1}AB = O_{n\times n}$

Since B is invertible. $B\cdot B^{-1} = B^{-1}B = I$

$A = B\cdot(B^{-1}AB)\cdot B^{-1} = B\cdot O_{n\times n}\cdot B^{-1} = O_{n\times n}$

Thus $N(\Phi) = \{O_{n\times n}\}$

- Φ is surjective

$\forall A \in M_{n\times n}(F)$. $\exists\ BAB^{-1} \in M_{n\times n}(F)$

s.t. $\Phi(BAB^{-1}) = B^{-1}\cdot(BAB^{-1})\cdot B = A$

Thus $R(\Phi) = M_{n\times n}(F)$.

or use $\dim(M_{n\times n}(F)) = \dim(N(\Phi)) + \dim(R(\Phi))$

In all, Φ is an isomorphism.

2. Consider a linear transformation $T : V \longrightarrow W$. Prove or disprove the following.

(a) If T has a right inverse, must it have a left inverse?

(b) If T has a left inverse, must it have a right inverse?

(c) If T has both a left and a right inverse, must it be invertible? (That is, must the left and right inverse be the same?)

(d) If T has a unique right inverse S, is T necessarily invertible? (Hint. Consider $ST + S - I$.)

(a) No.

$$T : F^\infty \longrightarrow F^\infty$$
$$(a_1, a_2, \cdots) \longmapsto (a_2, a_3, \cdots)$$
$$U : F^\infty \longrightarrow F^\infty$$
$$(a_1, a_2, \cdots) \longmapsto (0, a_1, a_2, \cdots)$$

Then $T \circ U = I$, T has a right inverse.

Suppose T has left inverse F then $F \circ T = I$.

$$\begin{cases} F \circ (T \circ U)((a_1, a_2, \cdots)) = F \circ I((a_1, a_2, \cdots)) \\ \qquad = F((a_1, a_2, \cdots)) \\ (F \circ T) \circ U((a_1, a_2, \cdots)) = I \circ U((a_1, a_2, \cdots)) \\ \qquad = U((a_1, a_2, \cdots)) \end{cases}$$

Thus $U = F$

but $U \circ T((1, 0, 0, \cdots)) = U((0, 0, \cdots)) = (0, 0, \cdots)$

i.e. $U \circ T = I$

contradiction. so T has no left inverse.

(b) No.

Consider
$$T : F^\infty \longrightarrow F^\infty$$
$$(a_1, a_2, \cdots) \longmapsto (0, a_1, a_2, \cdots)$$
$$U : F^\infty \longrightarrow F^\infty$$
$$(a_1, a_2, \cdots) \longmapsto (a_2, a_3, \cdots)$$

Then $U \circ T = I$ i.e. T has left inverse U.

By similar argument in (a), we know that T does not have a right inverse.

(c) if T has a left inverse U and a right inverse S

Then $U = U \circ I = U \circ (T \circ S) = (U \circ T) \circ S = I \circ S = S$

(d) T has a unique S. Then $T \circ S = I$.

$$
\begin{aligned}
T \circ (S \circ T + S - I) &= T \circ S \circ T + T \circ S - T \\
&= I \circ T + T \circ S - T \\
&= I
\end{aligned}
$$

Hence $S = S \circ T + S - I$

Then $S \circ T = I$, S is also the left inverse of T

T is invertible.

2. Let $g_0(x) = x + 1$. Let $T : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ and $U : P_3(\mathbb{R}) \longrightarrow \mathbb{R}^3$ be defined by

$$T(f(x)) = f'(x)g_0(x) + \int_0^x f(t)dt \text{ and } U(h(x)) = (h(0), h(1), h'(1))^T$$

Let α, β, γ be the standard ordered bases for $P_2(\mathbb{R}), P_3(\mathbb{R}), \mathbb{R}^3$ respectively.

(a) Compute $[T]_\alpha^\beta$, $[U]_\beta^\gamma$, $[U]_\beta^\gamma[T]_\alpha^\beta$ and $[UT]_\alpha^\gamma$.

(b) Let $h_0(x) = 1 - 2x - x^2 + x^3$, compute $[h_0(x)]_\beta$, $[U]_\beta^\gamma[h_0(x)]_\beta$ and $[U(h_0(x))]_\gamma$.

Solution.

$\alpha = \{1, x, x^2\}$ $\quad \beta = \{1, x, x^2, x^3\}$ $\quad \gamma = \{e_1\ e_2\ e_3\}$

(a) $T(1) = x$, $\quad T(x) = 1 + x + \frac{1}{2}x^2$, $\quad T(x^2) = 2x + 2x^2 + \frac{1}{3}x^3$

$U(1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\quad U(x) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\quad U(x^2) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\quad U(x^3) = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

$$[T]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \qquad [U]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$[U]_\beta^\gamma \cdot [T]_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \frac{5}{2} & \frac{13}{3} \\ 1 & 2 & 7 \end{pmatrix}$$

$$[UT]_\alpha^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \frac{5}{2} & \frac{13}{3} \\ 1 & 2 & 7 \end{pmatrix} \ ?$$

(b) $$[h_0(x)]_\beta = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

$$[U]_\beta^\gamma [h_0(x)]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$U(h_0) = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$[U(h_0(x))]_\gamma = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$