Theorem 1.10 (Replacement Theorem). *Let* V *be a vector space that is generated by a set* G *containing exactly* n *vectors, and let* L *be a linearly independent subset of* V *containing exactly* m *vectors. Then* $m \leq n$ *and there exists a subset* H *of* G *containing exactly* $n - m$ *vectors such that* $L \cup H$ *generates* V*.*

Finite-dim.

Sec. 1.7: Q7 Prove the following generalization of the replacement theorem. Let β be a basis for a vector space V, and let S be a linearly independent subset of V. There exists a subset S_1 of β such that $S \cup S_1$ is a basis for V.

Zorn's Lemma

Proof: Let $\mathcal{F}$ denotes the family of linearly independent subsets of $\beta \cup S$ that contain S.

Claim: For any chain $\mathcal{C}$ in $\mathcal{F}$, there exists a member U of $\mathcal{F}$ such that U contains all members of $\mathcal{C}$.

Let U be the union of all members of $\mathcal{C}$.

Then U contains all members of $\mathcal{C}$

$\forall\ u_1 \cdots u_n \in U$, $\exists\ A_i \in \mathcal{C}$ st $u_i \in A_i$. $i=1,\cdots,n$

Since $\mathcal{C}$ is a chain, one of these sets, say A_k, contains all the others. So $u_i \in A_k$ $i=1,\cdots,n$

Since $\{u_1,\cdots,u_n\}$ is a subset of lin. ind. A_k.

we have $\{u_1 \cdots u_n\}$ is lin. ind.

Since $\{u_1 \cdots u_n\}$ is chosen arbitrarily in U.

we have U is lin. ind.

Besides $S \subset U \subset \beta \cup S$.

Thus $U \in \mathcal{F}$.

The maximal principle implies that $\mathcal{F}$ contains a maximal element α. easily seen to be a maximal linearly independent subset of $\beta \cup S$ that contains S.

Let $S_1 = \alpha \setminus S \subset (\beta \cup S) \setminus S \subset \beta$

Then $\alpha = S_1 \cup S$ is a basis for V. (By Thm 1.12)

Q: Find ALL linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.
s.t. $R(T) = N(T)$.

A: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

① $T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow e=f=0$. so $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

② $\begin{cases} R(T) = N(T) \\ 2 = \dim(\mathbb{R}^2) = \dim(R(T)) + \dim(N(T)) \end{cases}$

$\Rightarrow \dim(R(T)) = \dim(N(T)) = 1$

Case 1: $R(T) = N(T) = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} t \mid t \in \mathbb{R} \right\}$

- $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in N(T)$. $T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e $\begin{pmatrix} a + 2b \\ c + 2d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $a = -2b$ $c = -2d$.

i.e. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

- $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \in R(T)$.

$\exists t \in \mathbb{R}$. $\begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot t$

$\Leftrightarrow$ $2(-2bx + by) = -2dx + dy$

$\Leftrightarrow$ $2b(-2x + y) = d(-2x+y)$ $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}$

$\Leftrightarrow$ $2b = d$ $\therefore T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -2b & b \\ -2^2 b & 2b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $b \neq 0$

Case 2: $R(T) = N(T) = \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} t \mid t \in \mathbb{R} \}$

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow b = d = 0$

$T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ cx \end{pmatrix}$

- $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$ $T(\begin{pmatrix} x \\ y \end{pmatrix}) \in R(T)$

$\exists t \in \mathbb{R}$ $\begin{pmatrix} ax \\ cx \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$

i.e $ax \cdot 1 = cx \cdot 0 = 0$ $\forall x.$

$\therefore$ $a = 0.$

$T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ linear transformation ?
$f(x) \mapsto x f(x) + f'(x)$

find $N(T)$ and $R(T)$

Solution: let $\beta = \{1, x, x^2\}$ be a basis for $P_2(\mathbb{R})$
$\gamma = \{1, x, x^2, x^3\}$ be a basis for $P_3(\mathbb{R})$

$$\begin{cases} T(1) = x \cdot 1 + (1)' = x \\ T(x) = x \cdot x + x' = x^2 + 1 \\ T(x^2) = x \cdot x^2 + (x^2)' = x^3 + 2x \end{cases}$$

- $R(T)$. β is basis for source space. $P_2(\mathbb{R})$

$T(\beta)$ is a generating set of $R(T)$

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{\underline{x, x^2+1, x^3+2x}\})$$

$$\dim(R(T)) = 3 < 4 = \dim(P_3(\mathbb{R}))$$ T is NOT onto.

- $N(T) = \{ g \in P_2(\mathbb{R}) ; T(g) = \vec{0} \in P_3(\mathbb{R}) \}$

$$= \{ a_0 + a_1 x + a_2 x^2 \in P_2(\mathbb{R}) : T(a_0 + a_1 x + a_2 x^2) = \vec{0} \}$$

$$\vec{0} = T(a_0 + a_1 x + a_2 x^2) = a_0 \cdot T(1) + a_1 \cdot T(x) + a_2 \cdot T(x^2)$$

$$= a_0 \cdot x + a_1 \cdot (x^2 + 1) + a_2 (x^3 + 2x)$$

$$= a_1 \cdot 1 + (a_0 + 2a_2) \cdot x + a_1 x^2 + a_2 x^3$$

has a unique solution $\begin{cases} a_1 = 0 \\ a_0 + 2a_2 = 0 \\ a_1 = 0 \\ a_2 = 0 \end{cases} \Rightarrow a_0 = a_1 = a_2 = 0$

$$N(T) = \{ \vec{0} \in P_2(\mathbb{R}) \} = \{\vec{0}\}$$

$\dim(N(T)) = 0$
T is 1-1