1. HW See 1.5 Q13

**13.** Let V be a vector space over a field of characteristic not equal to two.

**(a)** Let $u$ and $v$ be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.

**(b)** Let $u$, $v$, and $w$ be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is linearly independent.

Proof: field.

($\Rightarrow$) We aim to prove if $\{u, v, w\}$ linearly independent then $\{u+v, u+w, v+w\}$ linearly independent.

Consider the equation: $a(u+v)+b(u+v)+c(v+w)=\vec{0}$ where $a, b, c \in F$

By rearrangement, we have $(a+b)u+(a+c)v+(b+c)w=\vec{0}$

Since $\{u, v, w\}$ linearly independent, we have $a+b=a+c=b+c=0$

Besides, the characteristic of F is not 2, therefore $2=1+1\neq 0$ and the multiplicative inverse of 2 exists, i.e. $\frac{1}{2}\in F$

So $a=\dfrac{(a+b)+(a+c)-(b+c)}{2}=\dfrac{0+0-0}{2}=0$ and $b=c=0$

which means $\{u+v, u+w, v+w\}$ linearly independent.

($\Leftarrow$) We aim to prove if $\{u+v, u+w, v+w\}$ linearly independent then $\{u, v, w\}$ linearly independent

Consider the equation: $au+bv+cw=\vec{0}$ where $a, b, c \in F$

We can rewrite it as $\frac{a+b-c}{2}(u+v)+\frac{a+c-b}{2}(u+w)+\frac{b+c-a}{2}(v+w)=\vec{0}$

Since $\{u+v, u+w, v+w\}$ linearly independent, we have

$$\frac{a+b-c}{2}=\frac{a+c-b}{2}=\frac{b+c-a}{2}=0$$

So $a=\frac{a+b-c}{2}+\frac{a+c-b}{2}=0$ and similarly $b=c=0$

which means $\{u, v, w\}$ linearly independent.

2. direct sum

Sec. 1.6 Q34

**34.** **(a)** Prove that if $W_1$ is any subspace of a finite-dimensional vector space V, then there exists a subspace $W_2$ of V such that $V = W_1 \oplus W_2$.

**(b)** Let $V = R^2$ and $W_1 = \{(a_1, 0) : a_1 \in R\}$. Give examples of two different subspaces $W_2$ and $W_2'$ such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$.

(a) By existence of basis, we can choose a basis $\beta_1 = \{u_i\}_1^k$ for $W_1$

Then extend it to $\beta = \{u_i\}_1^k \cup \{v_j\}_1^m$, a basis for V.

Let $\beta_2 = \{v_j\}_1^m$, define $W_2 := \text{span}(\beta_2)$

We claim that $V = W_1 \oplus W_2$

1) $\forall\, w \in W_1 \cap W_2$, $\exists\, a_1 \cdots a_k, b_1 \cdots b_m \in F$ s.t

$$w = \sum_1^k a_i u_i = \sum_1^m b_j v_j$$

Then $0 = w - w = \sum_1^k a_i u_i - \sum_1^m b_j v_j$ ✡

Since $\beta$ is a basis for V, thus it's lin. ind.

The ✡ has a unique solution $a_i = b_j = 0 \quad \forall i, j$

Therefore $w = \vec{0}$ and $W_1 \cap W_2 = \{\vec{0}\}$

2) $V = \text{span}(\{u_1 \cdots u_k, v_1 \cdots v_m\})$

$\overset{✡}{=} \text{span}(\{u_1 \cdots u_k\}) + \text{span}(\{v_1 \cdots v_m\})$

$= W_1 + W_2$

Please show ✡ by yourself.

(b) Any straight line passing through the origin other than the x-axis is a subspace $W_2$ s.t $\mathbb{R}^2 = W_1 \oplus W_2$

3. direct product v.s. direct sum.

direct sum: $W_1, W_2$ be two subspaces of $V$. over $F$.

$V = W_1 \oplus W_2$ iff $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\vec{0}\}$

direct product: $W_1, W_2$ be two vector spaces. over $F$

$W_1 \times W_2 = \{ (\vec{w}_1, \vec{w}_2) \mid \vec{w}_1 \in W_1, \vec{w}_2 \in W_2 \}$

e.g. $W_1 = P_n := \{ \vec{f} \mid \vec{f}$ is poly and $\deg(f) \le n \}$ over $\mathbb{R}$

$W_2 = M_{k \times k}$ over $\mathbb{R}$.

$W_1 \times W_2 = \{ (\vec{f}, \vec{m}) \mid \vec{f} \in W_1, \vec{m} \in W_2 \}$

$\beta_1 = \{ x^0, x, x^2, \cdots x^n \}$ be basis for $W_1$

$\beta_2 = \{ E_{ij} \mid 1 \le i, j \le k \}$ be basis for $W_2$.

$$E_{ij} = \begin{array}{cc} \begin{bmatrix} 0 & \cdots & & 0 \\ & & 1 & \\ 0 & & & 0 \end{bmatrix} & i\text{th} \\ \quad\; j\text{th} & \end{array}$$

let $\vec{0}_1 \in W_1$, $\vec{0}_2 \in W_2$ be the zeros.

Check $\beta = \{ (x^l, \vec{0}_2) \}_{l=0}^{n} \cup \{ (\vec{0}_1, E_{ij}) \}_{i,j=1}^{k}$ is basis for $W_1 \times W_2$

- $\text{span}(\beta) = W_1 \times W_2$

$\Rightarrow$ $\beta \subset W_1 \times W_2$, so $\text{span}(\beta) \subset W_1 \times W_2$

$\Leftarrow$ $\forall x \in W_1 \times W_2$, $\exists \vec{f} \in W_1$ and $\vec{m} \in W_2$ st $x = (\vec{f}, \vec{m})$

$\exists a_0 \cdots a_n \in \mathbb{R}$ st $\vec{f} = \sum_{l=0}^{n} a_l \cdot x^l$

$\exists b_{11} \cdots b_{kk} \in \mathbb{R}$ st $\vec{m} = \sum_{i,j=1}^{k} b_{ij} \cdot E_{ij}$

then $x = (\vec{f}, \vec{m}) = \left( \sum_{l=0}^{n} a_l \cdot x^l , \sum_{i,j=1}^{k} b_{ij} \cdot E_{ij} \right)$

$$= \sum_{l=0}^{n} \sum_{i,j=1}^{k} ( a_l x^l , b_{ij} E_{ij} )$$

$$= \sum_{l=0}^{n} \sum_{i,j=1}^{k} \left[ a_l ( x^l , \vec{0}_2 ) + b_{ij} ( \vec{0}_1 , E_{ij} ) \right]$$

$$\in \operatorname{span}(\beta)$$

- $\beta$ is L.I.

Consider

$$\sum_{l=0}^{n} a_l ( x^l , \vec{0}_2 ) + \sum_{i,j=1}^{k} b_{ij} ( \vec{0}_1 , E_{ij} ) = \vec{0} \in W_1 \times W_2$$

$a_l, b_{ij} \in \mathbb{R}$.

Then $\left( \sum_{l=0}^{n} a_l x^l , \vec{0}_2 \right) + \left( \vec{0}_1 , \sum_{i,j=1}^{k} b_{ij} E_{ij} \right) = \vec{0}$

i.e. $\left( \sum_{l=0}^{n} a_l x^l , \sum_{i,j=1}^{k} b_{ij} E_{ij} \right) = \vec{0} \in W_1 \times W_2$

Then $\sum_{l=0}^{n} a_l x^l = \vec{0}_1 \in W_1$

$\Rightarrow a_0 = \cdots = a_n = 0$

and $\sum b_{ij} E_{ij} = 0$

$\Rightarrow b_{ij} = 0.$

# 4. quotient space

**31.** Let W be a subspace of a vector space V over a field $F$. For any $v \in V$ the set $\{v\}+W = \{v+w : w \in W\}$ is called the **coset** of W **containing** $v$. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

**(a)** Prove that $v + W$ is a subspace of V if and only if $v \in W$.
**(b)** Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

Addition and scalar multiplication by scalars of $F$ can be defined in the collection $S = \{v + W : v \in V\}$ of all cosets of W as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$ and

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in F$.

**(c)** Prove that the preceding operations are well defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$ $\to$ addition is well-def

and

$$a(v_1 + W) = a(v_1' + W)$$ $\to$ scaler multiplication is well-def

for all $a \in F$.

**(d)** Prove that the set $S$ is a vector space with the operations defined in (c). This vector space is called the **quotient space of V modulo W** and is denoted by V/W.

(a) ($\Rightarrow$) If $v+W$ is a subspace of $V$.

Then $\vec{0} \in v+W$ i.e. $\exists\, w \in W$ s.t. $\vec{0} = v+w$

Thus $v = -w \in W$ since $W$ is a subspace.

($\Leftarrow$) If $v \in W$.

$\forall x \in v+W$, $\exists\, y \in W$ s.t. $x = v+y$

Since $v, y \in W$ and $W$ is a subspace.

$x = v+y \in W$ i.e. $v+W \subset W$

$\forall x \in W$. $x = v + x + (-v)$

$y := x + (-v) \in W$ since $W$ is a subspace.

Thus $\exists\, y \in W$ s.t. $x = v+y \in v+W$ i.e. $W \subset v+W$

In conclusion, $W = v+W$ if $v \in W$, thus $v+W$ is a subspace.

(b) proved by Prof. Lui

(c) ① if $v_1+W=v_1'+W$. then $v_1-v_1' \in W$

if $v_2+W=v_2'+W$. then $v_2-v_2' \in W$.

$\exists w_1, w_2 \in W$. s.t $v_1-v_1'=w_1$

$v_2-v_2'=w_2$

$(v_1+v_2)-(v_1'+v_2')=w_1+w_2 \in W$.

$\therefore$ $(v_1+W)+(v_2+W)=(v_1+v_2)+W=(v_1'+v_2')+W=(v_1'+W)+(v_2'+W)$

② $v_1+W=v_1'+W$ iff $v_1-v_1' \in W$

So $a(v_1-v_1') \in W$

$av_1-av_1' \in W$ iff $av_1+W=av_1'+W$

$a(v_1+W)=av_1+W=av_1'+W=a(v_1'+W)$

(d) check VS 01 - VS 08

e.g.

VS 07:

$a\cdot((v_1+W)+(v_2+W))$

$=a((v_1+v_2)+W)$

$=a(v_1+v_2)+W=(av_1+av_2)+W$

$=(av_1+W)+(av_2+W)$