MATH2048 Honours Linear Algebra II

Midterm Examination 1

Please show all your steps, unless otherwise stated. Answer all **five** questions.

1. Let $W_1 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : a_1 + a_2 - a_4 = 0,\ a_2 + a_3 = 0\}$ and $W_2 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : a_1 + a_2 + 2a_3 + a_4 = 0,\ a_2 - a_4 = 0\}$.

   (a) Find a basis β_1 for W_1 and a basis β_2 for W_2.

   (b) Compute $\dim(W_1 + W_2)$ and use it to determine whether or not $\mathbb{R}^4 = W_1 \oplus W_2$.

2. Let $p_0(x) = x + 1$. Consider the following mapping

$$T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$$
$$p(x) \mapsto \begin{pmatrix} p(0) & p'(1) \\ (p_0 \cdot p)'(0) & \int_0^1 p(t)dt \end{pmatrix}$$

   Let $\beta = \{1, x, x^2\}$ and $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be bases for $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ respectively.

   (a) Show that T is a linear transformation.

   (b) Compute $[T]_\beta^\gamma$. Please show your steps.

   (c) Use the rank–nullity theorem to determine whether T is one-to-one. Please explain your answer with details.

3. Let $V = \{\sum_{m=1}^{K} a_m \sin(mx) + \sum_{n=1}^{K} b_n \cos(nx) : a_m, b_n \in \mathbb{R} \text{ for } m, n = 1 \dots K\}$ be a vector space over $\mathbb{R}$. The addition and scalar multiplication are defined as $(af + g)(x) = af(x) + g(x)$ for any $f, g \in V$ and $a \in \mathbb{R}$.

   Given that $\beta = \{\sin(mx), \cos(nx)\}_{m,n=1}^{K}$ is a basis for V. Let $T : V \to V$ be defined as $T(f) := -f'' + f$, where f'' refers to the second order derivative of f.

   (a) Show that T is a linear transformation.

   (b) Show that T is an isomorphism.

4. Let $V = C([0, 1], \mathbb{R})$ be the vector space of real-valued continuous functions on $[0, 1]$.

   (a) Let $\Phi : V \to \mathbb{R}^k$ be a linear transformation. Define the induced linear transformation $\widetilde{\Phi} : V/N(\Phi) \to \mathbb{R}^k$ by: $\widetilde{\Phi}(v + N(\Phi)) = \Phi(v)$. Show that $\widetilde{\Phi}$ is an isomorphism if and only if Φ is onto.

   (b) Let W be a subspace of V defined as follows:

$$W = \{f \in V : f(0) = f(1/N) = f(2/N) = ... = f((N-1)/N)\}.$$

   Construct an isomorphism between V/W and $\mathbb{R}^k$, where $k = \dim(V/W)$. Deduce the dimension of V/W.

5. Let V be an infinite dimensional vector space over F. Suppose W is a proper subspace of V (that is, $W \subsetneq V$). Consider the family of subspaces:

$$\mathcal{F} := \{A \subset V : A \text{ is a subspace and } A \cap W = \{\mathbf{0}\}\}.$$

   (a) Using Zorn's lemma, prove that $\mathcal{F}$ contains a maximal element $\widetilde{W}$.

   (b) Prove that $V = W \oplus \widetilde{W}$.