Lecture 9:

Space of linear transformation

Prop: Let V and W be vector spaces over F.

Then: the set $\mathcal{L}(V,W)$ of all linear transformations from V to W is a vector space over F under the following operations: for linear $T, U : V \to W$,

we define: $(T+U) : V \to W$ by $(T+U)(\underset{\in V}{\vec{x}}) = T(\vec{x}) + U(\vec{x})$

and for any $a \in F$, we define $aT : V \to W$ by

$$(aT)(\underset{\in V}{\vec{x}}) = a\,T(\vec{x})$$

Thm: Let V and W be finite-dimensional vector spaces over F. with dimension n and m respectively. Let β and γ be the ordered bases for V and W respectively.

Then: the map $\Phi : \mathcal{L}(V,W) \longrightarrow M_{m\times n}(F)$ defined by $\Phi(T) = [T]_\beta^\gamma$ is an isomorphism.

Cor: $\dim(\mathcal{L}(V,W)) = \dim(V)\dim(W) = nm$.

Proof: Φ is linear : $\Phi(T+U) = [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

$= \Phi(T) + \Phi(U)$

$\Phi(aT) = [aT]_\beta^\gamma = a[T]_\beta^\gamma$

$= a\,\Phi(T)$.

Φ is bijective :

For any $A = (A_{ij}) \in M_{m\times n}(F)$, ~~want to show that~~

$\exists !\, T : V \to W$ such that $\cancel{\Phi(T) = [T]_\beta^\gamma = A}$.

$$T(\vec{v}_j) = \sum_{i=1}^{m} A_{ij}\,\vec{w}_i \quad \text{for } j = 1, 2, \ldots, m$$

$$\begin{pmatrix} | \\ [T(\vec{v}_j)]_\gamma \\ | \end{pmatrix}$$

$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, $\gamma = \{\vec{w}_1, \ldots, \vec{w}_m\}$

$\Downarrow$

$\therefore$ For any $A \in M_{m\times n}(F)$, $\exists !\, T : V \to W$ such that $\Phi(T) = A$. (Onto)

(1-1)

$\therefore$ Φ is bijective.

Def Let β be the ordered basis for an n-dimensional vector space V over F. The map $\phi_\beta : V \to F^n$, $\vec{x} \longmapsto [x]_\beta$ is called **standard representation of V with respect to β.**

Prop: ϕ_β is an isomorphism.

Given vector spaces V and W of dimension n and m, with ordered bases β and γ respectively. Then, for any $T: V \to W$ (linear), we have:

$$\begin{tikzcd}
\vec{v} \in V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \ni T(\vec{v}) := \vec{w} \arrow[d, "\phi_\gamma"] \\
{[\vec{v}]_\beta \in F^n} \arrow[r, "L_A"] & F^m \quad [\vec{w}]_\gamma = [T(\vec{v})]_\gamma
\end{tikzcd}$$

where $A = [T]_\beta^\gamma$

$$\Rightarrow \quad \phi_\gamma \circ T(\vec{v}) = L_A \circ \phi_\beta(\vec{v})$$

$$\Leftrightarrow \quad [T(\vec{v})]_\gamma = [T]_\beta^\gamma [\vec{v}]_\beta$$

Change of coordinates

Prop: Let β and β' be two ordered bases for a finite-dim. vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$.

$$V_{\beta'} \xrightarrow{I_V} V_{\beta}$$

Then: (a) Q is invertible

(b) For all $\vec{v} \in V$, $[\vec{v}]_{\beta} = Q[\vec{v}]_{\beta'}$

roof: (a) Since I_V is invertible, Q is invertible.

b) Let $\vec{v} \in V$. Then: $[\vec{v}]_{\beta} = [I_V(\vec{v})]_{\beta} = \underbrace{[I_V]_{\beta'}^{\beta}}_{= Q}[\vec{v}]_{\beta'}$

f: The matrix $Q = [I_V]_{\beta'}^{\beta}$ is called the change of coordinate matrix from β' to β.

Remark: To compute $Q = [I_V]_{\beta'}^{\beta}$,

if $\beta = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ and $\beta' = \{\vec{x}_1{}', \vec{x}_2{}', \dots, \vec{x}_n{}'\}$,

then:

$$Q = \begin{pmatrix} | & & \\ [I_V(\vec{x}_1{}')]_{\beta} & \cdots & \\ | & & \end{pmatrix}$$

$$= \begin{pmatrix} | & & | & \\ [\vec{x}_1{}']_{\beta} & \cdots & [\vec{x}_j{}']_{\beta} & \cdots \\ | & & | & \end{pmatrix}$$

Proposition: Let T be a linear operator on finite-dim V. Let β and β' be ordered bases of V. Suppose $Q = [I_V]_{\beta'}^{\beta}$.

Then:
$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q$$

$$\overset{\beta}{V} \xrightarrow{T} \overset{\beta}{V} \rightsquigarrow [T]_{\beta}$$

$$\overset{\beta'}{V} \xrightarrow{T} \overset{\beta'}{V} \rightsquigarrow [T]_{\beta'}$$

Proof:
$$
\begin{aligned}
Q[T]_{\beta'} = [I_V]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} &= [I_V \circ T]_{\beta'}^{\beta} \\
&= [T \circ I_V]_{\beta'}^{\beta} \\
&= [T]_{\beta}^{\beta}[I_V]_{\beta'}^{\beta} \\
&= [T]_{\beta}Q
\end{aligned}
$$

$$\underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V} \xrightarrow{T} \underset{\beta}{V}$$

Remark: A linear $T : V \to V$ is called linear operator.

Corollary: Let $A \in M_{n \times n}(F)$ and let $\gamma = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ be an ordered basis for F^n.

Then: $[L_A]_\gamma = Q^{-1} A Q$, $Q = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{pmatrix}$

$\Leftrightarrow [L_A]_\gamma = Q^{-1} [L_A]_\beta Q$

standard ordered basis.