

Lecture 9:

Space of linear transformation

Prop: Let V and W be vector spaces over F .

Then: the set $\mathcal{L}(V, W)$ of all linear transformations from V to W is a vector space over F under the following operations: for linear $T, U: V \rightarrow W$,

we define: $(T+U): V \rightarrow W$ by $(T+U)(\vec{x}) = T(\vec{x}) + U(\vec{x})$

and for any $a \in F$, we define $aT: V \rightarrow W$ by

$$(aT)(\vec{x}) = aT(\vec{x})$$

Thm: Let V and W be finite-dimensional vector spaces over F .
with dimension n and m respectively. Let β and γ be the
ordered bases for V and W respectively.

Then: the map $\bar{\Phi} : \mathcal{L}(V, W) \rightarrow M_{m \times n}(F)$ defined
by $\bar{\Phi}(T) = [T]_{\beta}^{\gamma}$ is an isomorphism.

Cor: $\dim(\mathcal{L}(V, W)) = \dim(V) \dim(W) = nm$.

Proof: Φ is linear: $\Phi(T+U) = [T+U]_{\beta}^{\gamma} = [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}$
 $\Phi(aT) = [aT]_{\beta}^{\gamma} = a [T]_{\beta}^{\gamma} = \Phi(T) + \Phi(U)$
 $= a \Phi(T)$.

Φ is bijective:

For any $A = (A_{ij}) \in M_{m \times n}(F)$, ~~want to show that~~
 $\exists ! T: V \rightarrow W$ such that ~~$\Phi(T) = [T]_{\beta}^{\gamma} = A$.~~

$$T(\vec{v}_j) = \sum_{i=1}^m A_{ij} \vec{w}_i \quad \text{for } j=1, 2, \dots, m$$

$$\beta = \{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}, \quad \gamma = \{ \vec{w}_1, \dots, \vec{w}_m \}$$

\therefore For any $A \in \hat{M}_{m \times n}(F)$, $\exists ! T: V \rightarrow W$ such that $\Phi(T) = A$. (Onto)

$\therefore \Phi$ is bijective.

Def Let β be the ordered basis for an n -dimensional vector space V over F . The map $\phi_\beta: V \rightarrow F^n$, $\vec{x} \mapsto [x]_\beta$ is called **standard representation of V with respect to β** .

Prop: ϕ_β is an isomorphism.

Given vector spaces V and W of dimension n and m , with ordered bases β and γ respectively. Then, for any $T: V \rightarrow W$ (linear), we have:

$$\begin{array}{ccc}
 \vec{v} \in V & \xrightarrow{T} & W \ni T(\vec{v}) := \vec{w} \\
 \downarrow \phi_\beta & & \downarrow \phi_\gamma \\
 [\vec{v}]_\beta \in F^n & \xrightarrow{L_A} & F^m \quad [\vec{w}]_\gamma = [T(\vec{v})]_\gamma
 \end{array}$$

where $A = [T]_{\gamma}^{\beta}$

$$\begin{aligned}
 &\Rightarrow \phi_\gamma \circ T(\vec{v}) = L_A \circ \phi_\beta(\vec{v}) \\
 &\Leftrightarrow [T(\vec{v})]_\gamma = [T]_{\gamma}^{\beta} [\vec{v}]_\beta
 \end{aligned}$$

Change of coordinates

Prop: Let β and β' be two ordered bases for a finite-dim. vector space V , and let $Q = [I_V]_{\beta'}^{\beta}$. $V_{\beta'} \xrightarrow{I_V} V_{\beta}$

Then: (a) Q is invertible

(b) For all $\vec{v} \in V$, $[\vec{v}]_{\beta} = Q[\vec{v}]_{\beta'}$

Proof: (a) Since I_V is invertible, Q is invertible.

(b) Let $\vec{v} \in V$. Then: $[\vec{v}]_{\beta} = [I_V(\vec{v})]_{\beta} = [I_V]_{\beta'}^{\beta} [\vec{v}]_{\beta'}$

f: The matrix $Q = [I_V]_{\beta'}^{\beta}$ is called the Q change of coordinate matrix from β' to β .

Remark: To compute $Q = [I_V]_{\beta'}^{\beta}$,

if $\beta = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ and $\beta' = \{\vec{x}'_1, \vec{x}'_2, \dots, \vec{x}'_n\}$,

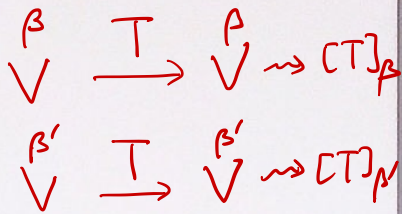
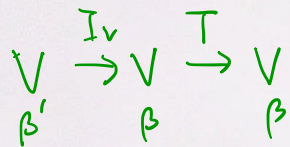
then:

$$Q = \begin{pmatrix} | & & | \\ [I_V(\vec{x}'_1)]_{\beta} & \dots & \\ | & & | \end{pmatrix}$$
$$= \begin{pmatrix} | & & | \\ [\vec{x}'_1]_{\beta} & \dots & [\vec{x}'_j]_{\beta} & \dots \\ | & & | \end{pmatrix}$$

Proposition: Let T be a linear operator on finite-dim V .
 Let β and β' be ordered bases of V . Suppose $Q = [I_V]_{\beta'}^{\beta}$.

Then: $[T]_{\beta'} = Q^{-1} [T]_{\beta} Q$

Proof: $Q [T]_{\beta'} = [I_V]_{\beta'}^{\beta} [T]_{\beta'} = [I_V \circ T]_{\beta'}^{\beta}$
 $= [T \circ I_V]_{\beta'}^{\beta}$
 $= [T]_{\beta}^{\beta} [I_V]_{\beta'}^{\beta}$
 $= [T]_{\beta} Q$



Remark: A linear $T: V \rightarrow V$ is called linear operator.

Corollary: Let $A \in M_{n \times n}(F)$ and let $\gamma = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ be an ordered basis for F^n .

$$\text{Then: } [L_A]_\gamma = Q^{-1} A Q, \quad Q = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \dots & \vec{x}_n \\ | & | & & | \end{pmatrix}$$

$$\Leftrightarrow [L_A]_\gamma = Q^{-1} [L_A]_\beta Q$$

\uparrow
standard
ordered
basis.