Lecture 8:

Recall: Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively. Let $T: V \to W$ be linear. Then: for any $\vec{u} \in V$, we have:

$$\underbrace{[\,T(\underset{\substack{\uparrow \\ W}}{\vec{u}})\,]_\gamma}_{\text{Lin. Transf.}} = \underbrace{[\,T\,]_\beta^\gamma \, [\,\vec{u}\,]_\beta}_{\text{Matrix multiplication}}$$

<u>Example:</u> $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(A) \stackrel{\text{def}}{=} A^T + 2A.$$

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Let $B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$.

$$[T(B)]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} [B]_\beta = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 4 \\ 0 \end{pmatrix}$$

$$T(B) = \begin{pmatrix} 9 & 5 \\ 4 & 0 \end{pmatrix}$$

Invertibility and Isomorphism

Definition: Let V and W be vector spaces and let $T: V \to W$ be linear. We can say T is invertible if it is bijective (1-1 and onto) so that $\exists\, T^{-1}: W \to V$ such that:

$$T^{-1} \circ T = I_V \quad \text{and} \quad T \cdot T^{-1} = I_W$$

- If V and W are of equal finite-dimensions, then $T: V \to W$ is invertible iff $\operatorname{rank}(T) = \dim(V)$.

- The inverse $T^{-1}: W \to V$ of an invertible linear transformation $T: V \to W$ is linear.

- Suppose $T: V \to W$ is invertible. Then:

$$\dim(V) < +\infty \text{ iff } \dim(W) < +\infty$$

And in this case, $\dim(V) = \dim(W)$

Proposition: Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively.

Let $T: V \to W$ be linear transformation.

Then: T is invertible iff $[T]_\beta^\gamma$ is invertible.

Furthermore,

$$\underbrace{[T^{-1}]_\gamma^\beta}_{\text{Matrix representation of } T^{-1}} = \underbrace{\left([T]_\beta^\gamma\right)^{-1}}_{\text{Inverse of matrix.}}$$

Proof: Suppose T is invertible. Then: $\dim(V) = \dim(W) = n$

Since $T \circ T^{-1} = I_W$, $I_n = [I_W]_\gamma = [T \circ T^{-1}]_\gamma$

$$\underset{\gamma}{W} \xrightarrow{T^{-1}} \underset{\beta}{V} \xrightarrow{T} \underset{\gamma}{W}$$

$$I_n = [T]_\beta^\gamma \, [T^{-1}]_\gamma^\beta$$

Similarly, $T^{-1} \circ T = I_V$. $I_n = [I_V]_\beta = [T^{-1} \circ T]_\beta$

$$I_n = [T^{-1}]_\gamma^\beta \, [T]_\beta^\gamma$$

$\therefore$ $[T]_\beta^\gamma$ is invertible and $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$.

Conversely, suppose $A := [T]_\beta^\gamma$ is invertible. $(\Rightarrow \dim(V) = \dim(W))$

$\because \dim(V) = \dim(W)$

$\therefore$ We only need to show T is one-to-one.

So, suppose $T(\vec{x}_1) = T(\vec{x}_2)$

$$\Leftrightarrow [T(\vec{x}_1)]_\gamma = [T(\vec{x}_2)]_\gamma$$

$$\Rightarrow \underbrace{[T]_\beta^\gamma}_{= A} [\vec{x}_1]_\beta = \underbrace{[T]_\beta^\gamma}_{= A} [\vec{x}_2]_\beta$$

$$\Rightarrow [\vec{x}_1]_\beta = [\vec{x}_2]_\beta \Rightarrow \vec{x}_1 = \vec{x}_2$$

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Corollary: Let V be a finite-dimensional vector space with ordered basis β. Let $T: V \to V$ be a linear transformation.

Then: T is invertible iff $[T]_\beta$ is invertible

Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

$[L_A]_\beta \leftarrow$ standard ordered basis

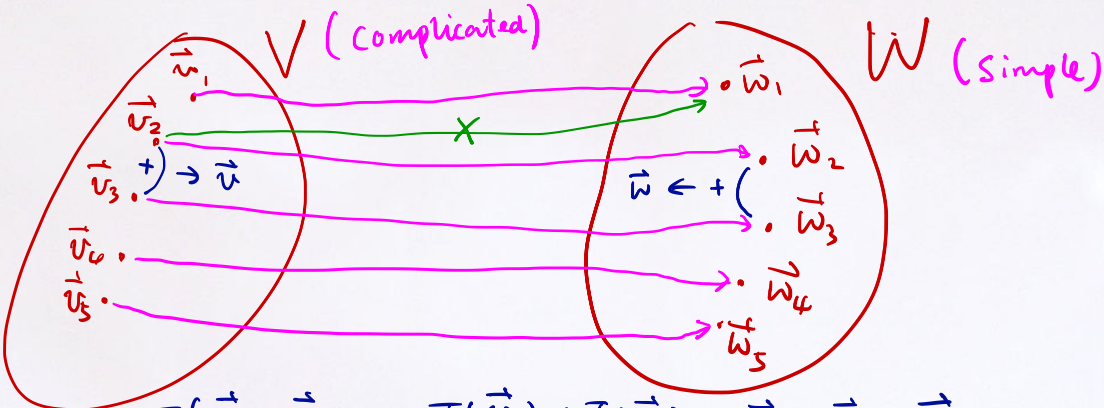
Corollary: Let $A \in M_{n\times n}(F)$. Then: A is invertible iff L_A is invertible. $(L_A)^{-1} = L_{A^{-1}}$

$$[(L_A)^{-1}]_\beta = ([L_A]_\beta)^{-1} = A^{-1} = [L_{A^{-1}}]_\beta$$

Definition: Let V and W be two vector spaces.

We say V is **isomorphic** to W if $\exists$ an invertible linear transformation $T: V \to W$.

In this case, T is called an **isomorphism** from V onto W.



$\vec{v}_2 + \vec{v}_3$

$\parallel$

$T(\vec{v}) = \vec{w}$??

$$T(\vec{v}_2 + \vec{v}_3) = T(\vec{v}_2) + T(\vec{v}_3) = \vec{w}_2 + \vec{w}_3 = \vec{w}$$

Thm: Let V and W be finite-dimensional vector spaces.

Then: V is isomorphic to W iff $\dim(V) = \dim(W)$.

Proof: $(\Rightarrow)$ This direction follows from previous Lemma.

$(\Leftarrow)$: Suppose $\dim(V) = \dim(W) \overset{\text{def}}{=} n$ and let

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be basis for V;

$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ be basis for W.

Then $\exists$ linear $T: V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$

for $i = 1, 2, \dots, n$.

By construction, T is onto and $\dim(V) = \dim(W)$.

So, T is one-to-one. $\therefore$ T is invertible.

<u>Corollary</u>: Let V be a vector space over F.

Then: V is isomorphic to F^n iff $\dim(V) = n$