Lecture 7:

Recall: <u>Matrix representation</u>

<u>Notation:</u> An ordered basis for a finite-dimensional vector space V is a basis for V endowed with a specific order.

$$\left(\text{e.g. } \mathbb{R}^2 \quad \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}_{=\beta_1} \neq \underbrace{\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}}_{=\beta_2} \text{ as ordered basis} \right)$$

<u>Definition:</u> Let V be a finite-dimensional vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be an ordered basis for V.

Then, $\forall \vec{x} \in V$, $\exists!\ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{u}_i$.

The coordinate vector of $\vec{x}$ relative to β, denoted as $[\vec{x}]_\beta$, is the column vector (F^n)

$$[\vec{x}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

Let V and W are finite-dimensional vector spaces with ordered bases $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ and $\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$ respectively.

(for V) (for W)

Let $T : V \to W$ be a linear transformation.

Then for each $1 \le j \le n$, $\exists\, a_{ij} \in F$ $1 \le i \le m$ such that

$$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij} \vec{w}_i \quad \text{for} \quad 1 \le j \le n,$$

W

Definition: With this notation as above, we call the matrix

$$A \overset{\text{def}}{:=} (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$$

the matrix representation of T in the ordered bases β and γ, and denoted it as $A = [T]_\beta^\gamma$.

If $V = W$ and $\beta = \gamma$, then we simply write

$$\cancel{[T]_\beta^\beta} \qquad [T]_\beta$$

$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for V

$\gamma = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ for W

$$[T]_\beta^\gamma = m\left[\underbrace{\begin{pmatrix} | & | & & | \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & | & & | \end{pmatrix}}_{n}\right.$$

$$\in$$

$$M_{m \times n}$$

Examples:

- Let $A \in M_{m\times n}(F)$. $L_A : F^n \to F^m$ defined by : $L_A(\underset{\in F^n}{\vec{x}}) \overset{def}{=} A\vec{x}$

Let β and γ be the standard bases for F^n and F^m resp.

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \dots \right\}$$

$$[L_A]_\beta^\gamma = \begin{pmatrix} | & & | \\ [A\vec{e}_1]_\gamma & \dots & [A\vec{e}_n]_\gamma \\ | & & | \end{pmatrix}$$

first col. of A

n^{th} col of A

$$A\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{first col of } A$$

$$= \begin{pmatrix} | & & | \\ 1^{st} \text{ col of } A & & n^{th} \text{ col of } A \\ | & & | \end{pmatrix} = A$$

- For $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined as $T(f(x)) = f'(x)$.

Let $\beta = \{1, x, x^2, \ldots, x^n\}$ be an ordered basis for $P_n(\mathbb{R})$

Let $\gamma = \{1, x, x^2, \ldots, x^{n-1}\}$ be an ordered basis for $P_{n-1}(\mathbb{R})$

$$[T]_\beta^\gamma = \begin{pmatrix} | & | & & | \\ [\underbrace{T(1)}_{=0}]_\gamma & [\overbrace{T(x)}^{=1}]_\gamma & \cdots & [\overbrace{T(x^n)}^{nx^{n-1}}]_\gamma \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Example: $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(A) \overset{\text{def}}{:=} A^T + 2A$$

tranpose

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$ — ordered basis

$$T(\beta) = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \right\}$$

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Example: $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by :

$$T(f) \stackrel{def}{=} \begin{pmatrix} f(0) & f(1) \\ 0 & f'(0) \end{pmatrix}$$

Consider ordered basis : $\beta = \{1, x, x^2\}$ for $P_2(\mathbb{R})$

$$\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$T(\beta) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_{4\times 3}$$

Composition of linear transformations and matrix multiplication

Thm: Let V and W be two vector spaces over the same field F.

And let $T: V \to W$ and $U: W \to Z$ be linear.

$$
\begin{tikzcd}
V_{\alpha}^{n} \arrow[r, "T"] \arrow[rr, bend left, "UT"] & W_{\beta}^{m} \arrow[r, "U"] & Z_{\gamma}^{p}
\end{tikzcd}
$$

(i) Then the composition $UT: V \to Z$ is linear.

(ii) If V, W, Z have ordered bases α, β, γ respectively,

then:
$$
\underset{M_{p\times n}}{[UT]_\alpha^\gamma} = \underbrace{\underset{M_{p\times m}}{[U]_\beta^\gamma}\,[T]_\alpha^\beta}_{\text{matrix multiplication}} \in M_{m\times n}
$$

(i) Let $\vec{x}, \vec{y} \in V$ and $a \in F$. Then:

$$UT(a\vec{x}+\vec{y}) = U(aT(\vec{x}) + T(\vec{y})) = aUT(x) + UT(\vec{y})$$

$\therefore$ UT is linear.

(ii) Suppose $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

$\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$

$\gamma = \{\vec{z}_1, \vec{z}_2, \dots, \vec{z}_p\}$

$$[UT]_\alpha^\gamma = (C_{ij})_{\substack{1 \le i \le p \\ 1 \le j \le n}}$$

$$\underset{M_{p\times m}(F)}{[U]_\beta^\gamma = A} \overset{\text{def}}{=} (a_{ik})_{\substack{1 \le i \le p \\ 1 \le k \le m}} \quad \text{means:} \quad U(\vec{w}_k) = \sum_{i=1}^{p} a_{ik} \vec{z}_i$$

$1 \le k \le m$

$$= \begin{pmatrix} & a_{1k} & \\ & a_{2k} & \\ & \vdots & \\ & a_{pk} & \end{pmatrix}$$

$\uparrow$ kth

$$[T]_\alpha^\beta = \underset{\in M_{m\times n}(F)}{B} \stackrel{\text{def}}{=} (b_{kj})_{\substack{1\le k\le m \\ 1\le j\le n}} \quad \text{means} \quad T(\vec{v}_j) = \sum_{k=1}^{m} b_{kj}\,\vec{w}_k$$

for $1 \le j \le n$

Then:

$$UT(\vec{v}_j) = U\left(\sum_{k=1}^{m} b_{kj}\,\vec{w}_k\right)$$

$$= \sum_{k=1}^{m} b_{kj}\, U(\vec{w}_k)$$

$$= \sum_{k=1}^{m} b_{kj}\left(\sum_{i=1}^{p} a_{ik}\,\vec{z}_i\right) = \sum_{i=1}^{p} \boxed{\left(\sum_{k=1}^{m} a_{ik}\, b_{kj}\right)} \vec{z}_i$$

(i, j)-entry of AB

So, $[UT]_\alpha^\gamma = AB = [U]_\beta^\gamma\,[T]_\alpha^\beta$

Example: Consider: $\mathcal{U} : \overset{\alpha}{P_n}(\mathbb{R}) \to \overset{\beta}{P_{n-1}}(\mathbb{R})$ defined by:

$$\mathcal{U}(f(x)) \overset{\text{def}}{=} f'(x)$$

Consider $T : \overset{\beta}{P_{n-1}}(\mathbb{R}) \to \overset{\alpha}{P_n}(\mathbb{R})$ defined by:

$$T(f(x)) \overset{\text{def}}{=} \int_0^x f(t)\,dt$$

Let α and β be standard ordered bases for $P_n(\mathbb{R})$ and $P_{n-1}(\mathbb{R})$ respectively.

$$[U]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & & 0 \\ \vdots & 0 & 2 & 0 & & 0 \\ \vdots & \vdots & 0 & 3 & & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & 0 & & n \end{pmatrix} \text{ and } [T]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 0 & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & \frac{1}{2} & 0 & & \vdots \\ \vdots & 0 & \frac{1}{3} & \cdots & \vdots \\ \vdots & \vdots & 0 & & 0 \\ 0 & 0 & \vdots & & \frac{1}{n} \\ & & 0 & & \end{pmatrix}$$

$$\text{So, } [UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha = \underset{\text{matrix}}{\text{Identity}} = [I_{P_{n-1}(\mathbb{R})}]_\beta$$

<u>Corollary:</u> Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively. Let $T: V \to W$ be linear. Then: for any $\vec{u} \in V$, we have:

$$[\underbrace{T(\vec{u})}_{\substack{W \\ \text{Lin. Transf.}}}]_\gamma = \underbrace{[T]_\beta^\gamma \, [\vec{u}]_\beta}_{\text{Matrix multiplication}}$$

<u>Proof:</u> Fix $\vec{u} \in V$ and consider two linear transformations:

$$f: \overset{\alpha}{F} \to \overset{\beta}{V} \qquad\qquad g: F \to \overset{\gamma}{W}$$

defined by   defined by

$$f(\underset{F}{a}) = a\vec{u} \in V \qquad\qquad g(a) = a\,T(\vec{u}) \in W$$

f and g are linear transformations. Also, $g = T \circ f$.

Let $\alpha = \{1\}$ be the standard ordered basis for F.

$$[T(\vec{u})]_\gamma = [g(1)]_\gamma = \underset{\underset{T\circ f}{\shortparallel}}{[g]_\alpha^\gamma} = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta$$

$$= [T]_\beta^\gamma [\vec{u}]_\beta$$

$$\begin{tikzcd} \overset{\alpha}{F} \arrow[r, "f"] \arrow[rr, bend right, "g = T\circ f"'] & \overset{\beta}{V} \arrow[r, "T"] & \overset{\gamma}{W} \end{tikzcd}$$

$$\begin{tikzcd} \overset{\alpha=\{1\}}{F} \arrow[r, "g"] & \overset{\gamma}{W} \end{tikzcd}$$

$$[g]_\alpha^\gamma = \cancel{\begin{pmatrix} | \\ [g(\vec{v}_j)]_\gamma \\ | \end{pmatrix}} = \begin{pmatrix} | \\ [g(1)]_\gamma \\ | \end{pmatrix}$$

Invertibility and Isomorphism

Definition: Let V and W be vector spaces and let $T: V \to W$ be linear. We can say T is invertible if it is bijective (1-1 and onto) so that $\exists\, T^{-1}: W \to V$ such that:

$$T^{-1} \circ T = I_V \quad \text{and} \quad T \cdot T^{-1} = I_W$$

Remark: If V and W are of equal finite-dimensions, then $T: V \to W$ is invertible iff $\operatorname{rank}(T) = \dim(V)$.

$$\underbrace{\dim(R(T)) \qquad \dim(W)}_{T \text{ is onto}}$$

Proposition: The inverse $T^{-1}: W \to V$ of an invertible linear transformation $T: V \to W$ is linear.

Proof: Let $\vec{y}_1, \vec{y}_2 \in W$ and $c \in F$.

$\because$ T is bijective $\therefore \exists!$ $\vec{x}_1 \in V$ and $\vec{x}_2 \in V$ such that

$$T(\vec{x}_1) = \vec{y}_1 \quad \text{and} \quad T(\vec{x}_2) = \vec{y}_2$$

So,
$$\begin{aligned}
T^{-1}(c\vec{y}_1 + \vec{y}_2) &= T^{-1}(c\,T(\vec{x}_1) + T(\vec{x}_2)) \\
&= T^{-1}(T(c\vec{x}_1 + \vec{x}_2)) \\
&= c\vec{x}_1 + \vec{x}_2 \\
&= c\,T^{-1}(\vec{y}_1) + T^{-1}(\vec{y}_2)
\end{aligned}$$

$\therefore$ T^{-1} is linear.

<u>Example</u>: 1. Let $A \in M_{n\times n}(F)$ is invertible.

Then: $L_A : F^n \to F^n$ defined by $L_A(\vec{x}) = A\vec{x}$.

is invertible and the inverse of L_A is:

$$(L_A)^{-1} = L_{A^{-1}}$$

2. If $T: V \to W$ and $U: W \to Z$ are invertible linear transformations, then: $U \circ T$ is also invertible and

$$(U \circ T)^{-1} = T^{-1} U^{-1}$$

$$\underbrace{\left(T^{-1} \underbrace{\cancel{U^{-1} \circ U}} \circ T \right)}_{Id}$$

Lemma: Suppose $T: V \to W$ is invertible.

Then: $\dim(V) < +\infty$ iff $\dim(W) < +\infty$

And in this case, $\dim(V) = \dim(W)$

Proof: Suppose $\dim(V) = n < +\infty$ and $\beta = \{\vec{x}_1, \ldots, \vec{x}_n\}$ is a basis for V. Then: $W = R(T) = \operatorname{span}\{T(\beta)\}$

$$= \operatorname{span}\underbrace{\{T(\vec{x}_1), \ldots, T(\vec{x}_n)\}}_{n \text{ elements}}$$

$\therefore \dim(W) \leq n = \dim(V) < +\infty$

Apply the same argument to T^{-1} to show that

$$\dim(V) \leq \dim(W)$$

In particular, if $\dim(V) < +\infty$ and $\dim(W) < +\infty$ then: $\dim(V) \leq \dim(W)$ and $\dim(W) \leq \dim(V) \iff \dim(V) = \dim(W)$