Lecture 3

Direct product space

Definition: Let V_1 and V_2 are vector spaces over F. Define:

$$V_1 \times V_2 = \{(\vec{x}, \vec{y}) : \vec{x} \in V_1 \text{ and } \vec{y} \in V_2\}$$

(called the direct product of V_1 and V_2)

Define:
- $(\vec{x}_1, \vec{y}_1) + (\vec{x}_2, \vec{y}_2) = (\vec{x}_1 + \vec{x}_2, \vec{y}_1 + \vec{y}_2)$ for $\forall \vec{x}_1, \vec{x}_2 \in V_1$, $\vec{y}_1, \vec{y}_2 \in V_2$
- $a(\vec{x}, \vec{y}) = (a\vec{x}, a\vec{y})$ for $\forall a \in F$, $\vec{x} \in V_1$, $\vec{y} \in V_2$.

Then: $V_1 \times V_2$ forms a vector space over F.

Exercise: Check.

Theorem: $\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2)$

Proof: (Idea) Let $\beta_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ = basis for V_1.
$\beta_2 = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ = basis for V_2.

Then: $\{(\vec{v}_1, \vec{0}_2), \ldots, (\vec{v}_n, \vec{0}_2), (\vec{0}_1, \vec{w}_1), \ldots, (\vec{0}_1, \vec{w}_m)\}$ forms a basis for $V_1 \times V_2$ (where $\vec{0}_1$ is the zero in V_1
$\vec{0}_2$ is the zero in V_2]

[Check! Exercise)

Remark: • For finite-dimensional vector space, direct product can be considered as direct sum.

Consider $X = Y_1 \times Y_2$ where $\dim(Y_1) < \infty$, $\dim(Y_2) < \infty$.

Let $X_1 = \{(\vec{y}_1, \vec{0}_2) : \vec{y}_1 \in Y_1\}$

$X_2 = \{(\vec{0}_1, \vec{y}_2) : \vec{y}_2 \in Y_2\}$

subspaces of X.

(where $\vec{0}_1$ = zero vector in Y_1

$\vec{0}_2$ = zero vector in Y_2)

Then: $X = X_1 \oplus X_2$

$= Y_1 \times Y_2$

Remark: • Consider $X = \mathbb{R}^\infty = \{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$

Then: $\dim(X) = \infty$.

Consider: $X_1 = \{(a_1, 0, 0, \dots,) : a_1 \in \mathbb{R}\}$;

$X_2 = \{(0, a_2, 0, \dots) : a_2 \in \mathbb{R}\}$;

$\vdots$

$X_i = \{(0, 0, \dots, a_i, 0 \dots 0) : a_i \in \mathbb{R}\}$

$\vdots$

Define: $X_1 \oplus X_2 \oplus X_3 \oplus \dots = \{\vec{x} = \vec{x}_{i_1} + \vec{x}_{i_2} + \dots + \vec{x}_{i_k} : \vec{x}_{i_j} \in X_{i_j},\ k \in \mathbb{N}\}$.

Then:

$X_1 \oplus X_2 \oplus \dots = \{(a_1, \dots, a_k, 0, 0, \dots) : a_i \in \mathbb{R},\ k \in \mathbb{N}\} \subsetneq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$

Direct product $\neq$ Direct Sum

- Direct product = collection of infinite sequence
- Direct sum = collection of finite sum / finite sequence.

Quotient Space

Definition: Let V be a vector space over F and let W be a subspace of V. Let $\vec{v} \in V$. Define:

$$\vec{v} + W = \{ \vec{v} + \vec{w} : \vec{w} \in W \}$$

$\vec{v} + W$ is called a coset of W in V.

Remark: $\vec{v} \in \vec{v} + W$.

Definition: The set V/W (called V mod W), is the set defined by $V/W = \{ \vec{v} + W : \vec{v} \in V \}$

(collection of cosets of W in V)

Proposition: Let $\vec{v}, \vec{v}' \in V$. Then: $\vec{v} + W = \vec{v}' + W$ iff $\vec{v} - \vec{v}' \in W$.

Proof: $(\Rightarrow)$ Let $\vec{v} + W = \vec{v}' + W$.

$\because$ $\vec{v} \in \vec{v} + W = \vec{v}' + W$. $\therefore$ $\vec{v} = \vec{v}' + \vec{w}$ for some $\vec{w} \in W$

$\therefore$ $\vec{v} - \vec{v}' = \vec{w} \in W$.

$(\Leftarrow)$ Suppose $\vec{v} - \vec{v}' \in W$.

Let $\vec{w} = \vec{v} - \vec{v}'$. Then: $v = \vec{v}' + w$, for some $\vec{w} \in W$.

$\therefore$ $\vec{v} + W \subset \vec{v}' + W$. Similarly, $\vec{v}' = \vec{v} + \vec{w}'$ for some $\vec{w}' \in W$.

$\Rightarrow \vec{v}' + W \subset \vec{v} + W$.

Definition: Define:

$$(\vec{v} + W) + (\vec{v}' + W) \overset{\text{def}}{:=} (\vec{v} + \vec{v}') + W \quad \text{(addition)}$$

$$a \cdot (\vec{v} + W) \overset{\text{def}}{:=} a\vec{v} + W \quad \text{(Scalar multiplication)}$$

Proposition: Suppose $\vec{v} + W = \vec{v}' + W$. Then: for any $\vec{v}'' + W \in V/W$.

- $(\vec{v} + W) + (\vec{v}'' + W) = (\vec{v}' + W) + (\vec{v}'' + W)$
- $a \cdot (\vec{v} + W) = a \cdot (\vec{v}' + W)$ for any $a \in F$.

Proof: Homework!

Remark: Addition and scalar multiplication are well-defined.

Theorem: With addition and scalar multiplication defined above, V/W is a vector space over F, called the quotient space.

Proof: Homework!

Examples of quotient space

- Let $W = \{\vec{0}\}$. V/W is the (isomorphic) same as V.

  Let $W = V$. V/V is the same as $\{\vec{0}\}$.

  $$\vec{v} + W = \vec{v}' + W \text{ iff } \vec{v} - \vec{v}' \in W = \{\vec{0}\}$$
  $$\text{iff } \vec{v} - \vec{v}' = \vec{0}$$
  $$\text{iff } \vec{v} = \vec{v}'.$$

- Let $V = \mathbb{R}^2$. Let W be the y-axis.

  Recall: $(x, y) + W = (x', y') + W$ iff $(x, y) - (x', y') \in W$
  iff $x - x' = 0$
  or $x = x'$

  $\therefore$ a vector in V/W is determined by the x-coordinate.

• Let $V = F^\infty$ (infinite sequence)

Let $W \overset{\text{def}}{:=} \{(0, x_2, x_3, \dots) : x_i \in F\}$.

As above, two vectors in V/W are the same iff they have the same first coordinate.

$(x_1, x_2, \dots) + W = (x_1', x_2', \dots) + W$ iff $(x_1 - x_1', x_2 - x_2', \dots) \in W$

iff $x_1 - x_1' = 0$ iff $x_1 = x_1'$.

$\therefore$ V/W is the same (isomorphic) as F

<u>Remark:</u> Even V and W are infinite dimensional, V/W is one-dimensional!

Proposition: Suppose V is finite-dimensional. Then:

$$\dim(V/W) = \dim(V) - \dim(W).$$

Proof: Let $\{\vec{w}_1, \ldots, \vec{w}_n\}$ be a basis of W.

Extend it to a basis $\{\vec{w}_1, \ldots, \vec{w}_n, \vec{v}_1, \ldots, \vec{v}_k\}$ of V.

Then: $\dim(W) = n$, $\dim(V) = n + k$

We'll prove that $\{\vec{v}_1 + W, \ldots, \vec{v}_k + W\}$ forms a basis of V/W.

If so, we'll have: $\dim(V/W) = k = \underbrace{(n+k)}_{\dim(V)} - \underbrace{n}_{\dim(W)}$

Linear independence:

Suppose: $a_1(\vec{v}_1 + W) + \ldots + a_k(\vec{v}_k + W) = \vec{0} + W$

$$\Rightarrow (a_1\vec{v}_1 + \ldots + a_k\vec{v}_k) + W = \vec{0} + W$$

$$\therefore \quad a_1\vec{v}_1 + \dots + a_k\vec{v}_k \in W$$

$$\Rightarrow a_1\vec{v}_1 + \dots + a_k\vec{v}_k = b_1\vec{w}_1 + \dots + b_n\vec{w}_n \quad \text{for some } b_1, \dots, b_n \in F.$$

$$\Rightarrow a_1\vec{v}_1 + \dots + a_k\vec{v}_k - b_1\vec{w}_1 - \dots - b_n\vec{w}_n = \vec{0}$$

As $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_n\}$ is linearly independent,

$$a_1 = \dots = a_k = 0 \quad \text{and} \quad b_1 = \dots = b_n = 0.$$

$\therefore \{\vec{v}_1 + W, \dots, \vec{v}_k + W\}$ is linear independent.

<u>Span:</u> Let $\vec{v} + W \in V/W$.

Then: $\vec{v} = a_1\vec{w}_1 + \dots + a_n\vec{w}_n + b_1\vec{v}_1 + \dots + b_k\vec{v}_k$ for some a_i's and b_j's.

$$\Rightarrow \vec{v} + W = \Big(b_1\vec{v}_1 + \dots + b_k\vec{v}_k + \underbrace{a_1\vec{w}_1 + \dots + a_n\vec{w}_n}_{W}\Big) + W$$

$$= b_1(\vec{v}_1 + W) + \dots + b_k(\vec{v}_k + W)$$