Lecture 2:

Recap:

Definition: A vector space V is called <u>finite-dimensional</u> if it has a finite basis. The <u>dimension</u> of V, denoted as $\dim(V)$, is the number of vectors in a basis for V.

A vector space which is not finite-dimensional is called <u>infinite-dimensional</u>

Example:

- $\mathbb{C}^n$ is n-dimensional
- $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is infinite-dimensional

Example : $P_n(F)$ where $F = F_2$ (binary field)

Then: $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(F)$.

$\therefore$ $P_n(F)$ is $(n+1)$-dimensional.

# of elements in $P_n(F)$ is 2^{n+1}

$$\Big(\underset{\{0,1\}}{\underset{\cap}{a_0}} + \underset{\{0,1\}}{\underset{\cap}{a_1}} x + \ldots + \underset{\{0,1\}}{\underset{\cap}{a_n}} x^n$$

Finite-dimensional vector space with finitely many elements.

Direct Sum: Let U and W be subspaces of V.

Then: $U + W = \{ \vec{x} + \vec{y} : \vec{x} \in U \text{ and } \vec{y} \in W \}$ is also a subspace of V (Check!)

Definition: V is said to be the <u>direct sum</u> of U and W, denoted by $V = U \oplus W$ if $V = U + W$ and $U \cap V = \{\vec{0}\}$.

Lemma: $V = U \oplus W$ iff for $\forall \vec{v} \in V$, $\exists !$ vectors $\vec{u} \in U$ and $\vec{w} \in W$ $\ni$ $\vec{v} = \vec{u} + \vec{w}$

Proof: $(\Rightarrow)$ If $\vec{v} \in V$, then $\vec{v} = \vec{u} + \vec{w}$ for some $\vec{u} \in U$ and $\vec{w} \in W$ $(\because V = U \oplus W)$

For uniqueness, let $\vec{v} = \vec{u}_1 + \vec{w}_1 = \vec{u}_2 + \vec{w}_2$.

Then: $\vec{u}_1 - \vec{u}_2 = \vec{w}_2 - \vec{w}_1 \in U \cap W = \{\vec{0}\}$. $\therefore$ $\vec{u}_1 - \vec{u}_2 = \vec{0} \Rightarrow \vec{u}_1 = \vec{u}_2$

$\vec{w}_2 - \vec{w}_1 = \vec{0} \Rightarrow \vec{w}_1 = \vec{w}_2$

($\Leftarrow$) $V = U + W$ is obvious.

Now, let $\vec{z} \in U \cap W$. $\exists!\ \vec{u}$ and $\vec{w} \ni \vec{z} = \vec{u} + \vec{w}$.

Then: $\vec{z} = \underset{\in U}{\vec{z}} + \underset{\in W}{\vec{0}} = \underset{\in U}{\vec{0}} + \underset{\in W}{\vec{z}} \Rightarrow \vec{u} = \vec{0}$ and $\vec{w} = \vec{0}$

$\Rightarrow \vec{z} = \vec{0}$.

$\therefore\ U \cap W = \{\vec{0}\}$

Examples: $F = \mathbb{R}$

- $\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\} \oplus \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\} = \mathbb{R}^3$ is a direct sum.

- $\underbrace{\left\{ \begin{matrix} \text{Even functions} \\ f(-t) = f(t) \end{matrix} \right\}}_{= W_1} \oplus \underbrace{\left\{ \begin{matrix} \text{Odd functions} \\ f(t) = -f(-t) \end{matrix} \right\}}_{= W_2}$ is a direct sum.

$= \{\text{Space of all functions}\}$

if $g \in W_1 \cap W_2$, then: $f(t) = f(-t) = -f(t) \Rightarrow 2f(t) = 0 \Rightarrow f(t) = 0$

$\therefore W_1 \cap W_2 = \{0\}$

- $\{\text{Constant}\} \oplus \{\text{polynomials } p(t) : p(0) = 0\}$ is a direct sum

$= \{\text{Space of all polynomials}\}$

- $\left\{ \begin{matrix} \text{Symmetric matrices :} \\ A^T = A \end{matrix} \right\} \oplus \left\{ \begin{matrix} \text{Anti-symmetric matrices:} \\ A^T = -A \end{matrix} \right\}$ is a direct sum.

$= \{\text{Space of all matrices}\}$

Projection operators

Definition: Suppose $V = U \oplus W$. Define : $P : V \to U$ as follows :

For any $\vec{v} \in V$, write $\vec{v} = \vec{u} + \vec{w}$ where $\vec{u} \in U$ and $\vec{w} \in W$.

Then: define : $P(\vec{v}) = \vec{u}$

Remark:

1. P is well-defined
2. $P \circ P = P$

Definition V is said to be a direct sum of subspaces $U_1, U_2, \dots, U_k$, denoted as $V = U_1 \oplus U_2 \oplus \dots \oplus U_k$, if for $\forall \vec{v} \in V$, $\exists!$ vectors $\vec{u}_i \in U_i$ $(1 \le i \le k)$ $\ni$ $\vec{v} = \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_k$.

Remark:

- $U_1 \oplus \dots \oplus U_k = (\cdots ((U_1 \oplus U_2) \oplus U_3) \oplus \cdots \oplus U_k)$
- $V = U_1 \oplus U_2 \oplus \dots \oplus U_k$ iff :

  ① $V = U_1 + U_2 + \dots + U_k$

  ② $U_r \cap \sum_{i \neq r} U_i = \{\vec{0}\}$ for $1 \le r \le k$.

Dimension of direct sum

Theorem: Let V be a finite-dim vector space. $U_1, U_2, \ldots, U_m$ are subspaces of V. Then:

$$\dim(U_1 \oplus U_2 \oplus \ldots \oplus U_m) = \sum_{i=1}^{m} \dim(U_i)$$

Proof: Let β_i = basis of U_i for $i = 1, 2, \ldots, m$.

Let $\beta = \beta_1 \,\dot\cup\, \beta_2 \,\dot\cup \ldots \dot\cup\, \beta_m$ (disjoint union)

For $\forall \vec{v} \in U_1 \oplus \ldots \oplus U_m$, $\exists!\ \vec{u}_1 \in U_1, \vec{u}_2 \in U_2, \ldots, \vec{u}_m \in U_m \Rightarrow$

$$\vec{v} = \vec{u}_1 + \ldots + \vec{u}_m.$$

Each $\vec{u}_i$ can be written as a linear combination of elements in β_i.

$\therefore \operatorname{Span}(\beta) = U_1 \oplus \ldots \oplus U_m$

β is linear independent.

Let $\vec{0} = (a_1^1 u_1^1 + a_2^1 u_2^1 + \dots + a_{n_1}^1 u_{n_1}^1) + \dots + (a_1^m u_1^m + \dots + a_{n_m}^m u_{n_m}^m)$

$u_1^1 \in \beta_1$, $u_2^1 \in \beta_1$, $u_{n_1}^1 \in \beta_1$, $u_1^m \in \beta_m$, $u_{n_m}^m \in \beta_m$

Then: each $a_1^j u_1^j + \dots + a_n^j u_{n_j}^j = \vec{0}$ for $\forall j$

$\Rightarrow a_1^j = a_2^j = \dots = a_{n_j}^j = 0$ for $\forall j$.

$\therefore$ β is linear independent.

$\therefore$ β is a basis.

$\therefore \dim(U_1 \oplus \dots \oplus U_m) = |\beta| = \sum_{i=1}^{m} \dim(U_i)$.

Remark: In general,

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

(Homework!)