Lecture 21:

Recap

Main Theorem: (Jordan Decomposition Theorem)

Let $A \in M_{n\times n}(\mathbb{C})$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (distinct) with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Then:

(1) $\dim(K_{\lambda_i}) = m_i$

(2) $\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$

(3) Each K_{λ_i} has a basis $\beta_i = \gamma_{1,i} \cup \cdots \cup \gamma_{\ell,i}$ where every $\gamma_{m,i}$ is a cycle:

$$\gamma_{m,i} = \{(A-\lambda_i I)^{p-1}\vec{x}, (A-\lambda_i I)^{p-2}\vec{x}, \ldots, (A-\lambda_i I)\vec{x}, \vec{x}\}$$

Theoretical proof: $T: V \to V$ (fin-dim), char poly splits.
($\lambda_1, \lambda_2, \ldots, \lambda_k$ distinct eigenvalues)

Last time:

Claim 1: $(T - \lambda_i I)\big|_{K_{\lambda_j}} : K_{\lambda_j} \to K_{\lambda_j}$ is 1-1 if $i \neq j$.

Claim 2: $\dim(K_{\lambda_i}) \leq m_i$ = multiplicity of λ_i and
$K_{\lambda_i} = N((T - \lambda_i I)^{m_i})$

<u>Claim 3:</u> $V = K_{\lambda_1} + K_{\lambda_2} + \dots + K_{\lambda_k}$

<u>Pf:</u> By M.I. on k = # of distinct eigenvalues.

When $k=1$, let m = multiplicity of λ_1. Then, char poly of T " $(\lambda_1 - t)^m$.

By Caley-Hamilton Thm, $g(T) = (\lambda_1 I - T)^m = 0 \leftarrow$ zero transf.

$$\therefore \quad K_{\lambda_1} = N\left((T - \lambda_1 I)^m\right) = V$$

$\therefore$ Thm is true for $k=1$.

Assume that the thm is true for any lin. op. w/ fewer than k distinct eigenvalues.

Consider $T: V \to V$ with k distinct eigenvalues:

$$\lambda_1, \lambda_2, \dots, \lambda_k$$

Claim 4: Let $W = R\left((T-\lambda_k I)^{m_k}\right)$ ← multiplicity of λ_k

Then: ① $T|_W : W \to W$ is well-defined. (exercise)

② $T|_W$ has $k-1$ distinct eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$

③ $(T-\lambda_k I)^{m_k}\big|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$ is onto $(i<k)$

Assume Claim 4 is true.

$(m = m_k)$

Let $\vec{x} \in V$. Then: $(T-\lambda_k I)^m \vec{x} \in W$

By induction hypothesis, $\exists\, \vec{w}_i \in K'_{\lambda_i}$ = generalized eigenspace of λ_i of $T|_W$

such that

$$(T-\lambda_k I)^m \vec{x} = \underset{\substack{\uparrow \\ K'_{\lambda_1}}}{\vec{w}_1} + \underset{\substack{\uparrow \\ K'_{\lambda_2}}}{\vec{w}_2} + \dots + \vec{w}_{k-1}$$

Also,

$K'_{\lambda_i} \subseteq K_{\lambda_i}$ for $i<k$

$\because\ (T-\lambda_k I)^m\big|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$ is onto, then:

for each $\vec{w}_i \in K_{\lambda_i}$, $\exists\ \vec{v}_i \in K_{\lambda_i}$ such that $(T-\lambda_k I)^m(\vec{v}_i) = \vec{w}_i$

$$\therefore\ (T-\lambda_k I)^m(\vec{x}) = (T-\lambda_k I)^m(\vec{v}_1) + \dots + (T-\lambda_k I)^m(\vec{v}_{k-1})$$

$$\Longleftrightarrow\ (T-\lambda_k I)^m\Big(\vec{x} - \underset{K_{\lambda_1}}{\underset{\uparrow}{\vec{v}_1}} - \underset{K_{\lambda_2}}{\underset{\uparrow}{\vec{v}_2}} - \dots - \underset{K_{\lambda_{k-1}}}{\underset{\uparrow}{\vec{v}_{k-1}}}\Big) = \vec{0}$$

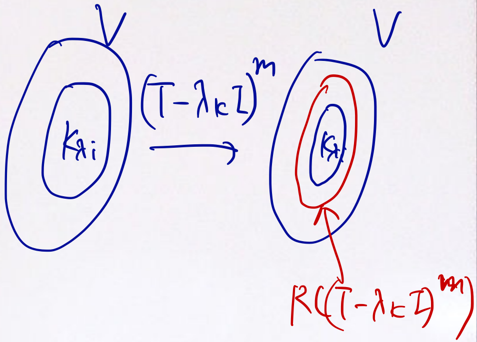
$$\therefore\ \overbrace{\vec{x} - \vec{v}_1 - \vec{v}_2 - \dots - \vec{v}_{k-1}}^{\vec{v}_k} \in N\big((T-\lambda_k I)^m\big) = K_{\lambda_k}$$

$$\therefore\ \vec{x} = \underset{K_{\lambda_1}}{\underset{\uparrow}{\vec{v}_1}} + \underset{K_{\lambda_2}}{\underset{\uparrow}{\vec{v}_2}} + \dots + \underset{K_{\lambda_{k-1}}}{\underset{\uparrow}{\vec{v}_{k-1}}} + \underset{K_{\lambda_k}}{\underset{\uparrow}{\vec{v}_k}}$$

Proof of Claim 4

Note: $(T-\lambda_k I)\big|_{K_{\lambda_i}}$ is 1-1 and onto (if $i<k$)

$\Rightarrow \quad (T-\lambda_k I)^m\big|_{K_{\lambda_i}}$ is also onto.



Also, $E_{\lambda_i} \subseteq K_{\lambda_i} \subseteq W = R((T-\lambda_k I)^m)$

$\therefore$ λ_i is an eigenvalue of $T|_W$ for $i<k$

$\therefore$ $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$ are eigenvalues of $T|_W$

Next, suppose λ_k is an eigenvalue of $T|_W$.

Suppose $T|_W(\vec{v}) = \lambda_k \vec{v}$ for $\vec{v} \neq \vec{0}$ and $\vec{v} \in W = R((T-\lambda_k I)^m)$

Write $\vec{v} = (T-\lambda_k I)^m(\vec{y})$.

$$\therefore \quad \vec{0} = (T-\lambda_k I)\vec{v} = (T-\lambda_k I)(T-\lambda_k I)^m(\vec{y})$$
$$= (T-\lambda_k I)^{m+1}(\vec{y})$$

$$\Rightarrow \quad \vec{y} \in K_{\lambda_k} = N((T-\lambda_k I)^m)$$

$$\therefore \quad \underbrace{(T-\lambda_k I)^m(\vec{y})}_{\vec{v}} = \vec{0}$$

Contradiction.

Claim 5: Let β_i = ordered basis of K_{λ_i}.

Then: $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$ is a disjoint union and a basis of V.

Pf Disjoint union: Let $\overset{\neq \vec{0}}{\vec{x}} \in \beta_i \cap \beta_j \ (i \neq j) \subseteq K_{\lambda_i} \cap K_{\lambda_j}$.

$$K_{\lambda_j} \ni (T - \lambda_i I)\Big(\overset{\neq \vec{0}}{\underset{\in K_{\lambda_j}}{\vec{x}}}\Big) \neq \vec{0} \qquad \left((T - \lambda_i I)\big|_{K_{\lambda_j}} \text{ is } 1\text{-}1 \right)$$

$$(T - \lambda_i I)^2 (\vec{x}) \neq \vec{0}$$

$$\vdots$$

$$(T - \lambda_i I)^p (\vec{x}) \neq \vec{0} \quad \text{for } \forall p.$$

$$\therefore \ \vec{x} \notin K_{\lambda_i} \quad (\text{Contradiction})$$

Basis: Let $\vec{x} \in V$. By claim 3,

$$\vec{x} = \vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_k \quad \text{where} \quad \vec{v}_i \in K_{\lambda_i}$$

$\vec{v}_1 \in K_{\lambda_1} = \mathrm{span}(\beta_1)$, $\quad \vec{v}_2 \in K_{\lambda_2} = \mathrm{span}(\beta_2)$, $\quad \vec{v}_k \in K_{\lambda_k} = \mathrm{span}(\beta_k)$

$\therefore V = \mathrm{span}(\beta)$

Let $q = |\beta|$. Then: $\dim(V) \le q$

Let $d_i = \dim(K_{\lambda_i})$. Then: $q = \sum_i d_i \le \sum_i m_i = \dim(V)$

$\therefore \dim(V) = q \Rightarrow \beta$ is a basis.

<u>Claim 6:</u> $\dim(K_{\lambda_i}) = m_i$

<u>Pf:</u>

$$\sum_i d_i = \sum_i m_i \Rightarrow \sum_i \overbrace{(m_i - d_i)}^{\geq 0} = 0$$

$$\sum_i d_i = \dim(V) = \sum_i m_i$$

$\Rightarrow m_i = d_i$ for $\forall i$.

$$V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$$

Claim 7: Let $\gamma_1 = \{ (T - \lambda I)^{m_1} \vec{v}_1, \dots, \vec{v}_1 \}$

$$\vdots$$

$$\gamma_q = \{ (T - \lambda I)^{m_q} \vec{v}_q, \dots, \vec{v}_q \}$$

If initial vectors are linearly independent, then:

$\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_q$ is disjoint union and it's linearly independent.

Pf: Disjoint union: exercise

Linear independence: Use M.I. on n = # of element in γ.

When $n=1$, trivial

Assume the thm is true for γ having less than n elements

When $|\gamma| = n$, let

$$\gamma_1' = \{ (T-\lambda I)^{m_1}\vec{v}_1, \dots, (T-\lambda I)\vec{v}_1 \}$$

$$\vdots$$

$$\gamma_q' = \{ (T-\lambda I)^{m_q}\vec{v}_q, \dots, (T-\lambda I)\vec{v}_q \}$$

Let $\gamma' = \gamma_1' \cup \gamma_2' \cup \dots \cup \gamma_q'$. $\therefore |\gamma'| = n-q$

Let $W = \mathrm{span}(\gamma)$. Let $U = (T-\lambda I)|_W : W \to W$.

Then: $R(U) = \mathrm{Span}(\gamma')$ (Check)

$\because$ initial vectors of γ_i's are L.I.

$\therefore$ γ' is L.I. (by induction hypothesis.)

$\therefore$ $\dim(R(U)) = |\gamma'| = n-q$

Also, $S = \{ (T-\lambda I)^{m_1} \vec{v}_1, \dots, (T-\lambda I)^{m_q} (\vec{v}_q) \} \subseteq N((T-\lambda I)|_W)$

$$\begin{array}{c} N((T-\lambda I)|_W) \\ \| \\ N(U) \end{array}$$

$\therefore \quad \dim(N(U)) \geq q$

$U = (T-\lambda I)|_W : W \to W$

$$\therefore \quad \begin{array}{c} n \\ \| \\ |\gamma| \end{array} \geq \dim(W) = \dim(R(U)) + \dim(N(U))$$

$$\geq n-q+q = n$$

$\therefore \quad \dim(W) = n, \quad |\gamma| = n \quad$ and $\quad \operatorname{span}(\gamma) = W$

$\therefore \quad \gamma$ is a basis of W and γ is L.I.

Claim 8: Suppose β = basis of V = disjoint union of cycles.

Then: (1) For each cycle γ in β, $W = \mathrm{span}(\gamma)$ is T-invariant and $[T|_W]_\gamma$ = Jordan block.

(2) β = JC basis for V,

Claim 9: Let λ = eigenvalue of T.

Then: K_λ has a basis β = union of disjoint cycles w.r.t. λ.

Pf: By M.I. on $n = \dim(K_\lambda)$

When $n = 1$, trivial.

Suppose the result is true for $\dim(K_\lambda) < n$.

When $\dim(K_\lambda) = n$. Let $U = (T - \lambda I)\big|_{K_\lambda} : K_\lambda \to K_\lambda$

Then: $\dim(R(U)) < \dim(K_\lambda) = n$

$\left(\because \ \dim(K_\lambda) = \dim(\underset{E_\lambda}{\underset{\cup\!\mid}{N(U)}}) + \dim(R(U)), \ \dim(E_\lambda) \geq 1 \right)$

Let $K_\lambda{}'$ = generalized eigenspace corresponding to λ of $T\big|_{R(U)} : R(U) \to R(U)$.

$R(U) = K_\lambda{}'$

By induction hypothesis, $\exists$ disjoint cycles $\gamma_1, \gamma_2, \dots, \gamma_q$ of $T|_{R(U)}$ $\ni$

$$\gamma = \bigcup_{i=1}^{q} \gamma_i \text{ is a basis for } R(U) = K_\lambda'$$

Let $\gamma_i = \{ \overbrace{(T|_{R(U)} - \lambda I)^{m_i}\vec{x}_i}^{w_i}, \dots, \vec{x}_i \}$

$\widehat{K_\lambda'} = R(U)$

Let $\vec{x}_i = U\vec{v}_i = (T - \lambda I)\vec{v}_i$, $\vec{v}_i \in K_\lambda$

Define: $\tilde{\gamma}_i = \{ \underbrace{(T - \lambda I)^{m_i+1}(\vec{v}_i)}_{\vec{w}_i}, \dots, (T - \lambda I)(\vec{v}_i), \vec{v}_i \}$

Note: $S = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$ is L.I. subset of E_λ

Extend S to a basis of $E_\lambda = \{\vec{w}_1, \dots, \vec{w}_q, \vec{u}_1, \vec{u}_2, \dots, \vec{u}_s\}$

By construction, $\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_q, \{\vec{u}_1\}, \{\vec{u}_2\}, \dots, \{\vec{u}_s\}$ are disjoint union of cycles $\ni$ initial vectors are L.I.

$\therefore \ \tilde{\gamma} = \bigcup_{i=1}^{q} \tilde{\gamma}_i \cup \{\vec{u}_1\} \cup \dots \cup \{\vec{u}_s\}$ is L. I. subset of K_λ,

Now, we show $\tilde{\gamma}$ is a basis of K_λ.

$U : K_\lambda \to K_\lambda$

Suppose $|\gamma| = r = \dim(R(U))$

Then: $|\tilde{\gamma}| = r + q + s$

$\because \ \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q, \vec{u}_1, \vec{u}_2, \dots, \vec{u}_s\}$ is a basis of E_λ

$$E_\lambda = N(U) = N((T - \lambda I)|_{K_\lambda})$$

$\therefore \ \dim(N(U)) = q + s.$

$\therefore \ \dim(K_\lambda) = \dim(R(U)) + \dim(N(U)) = r + q + s = |\tilde{\gamma}|$

$\therefore \ \tilde{\gamma}$ is a basis for K_λ

Claim 10: T has JCF.