Lecture 20

Jordan Canonical Form

Recall: Let $T: V \to V$ lin. operator on a fin-dim V (over F)

$T: V \to V$ is diagonalizable $\iff$

1. Char poly splits.
2. $\dim(E_{\lambda_i}) = m_i$ ← alg. multiplicity for all eigenvalues λ_i

(In general, $\dim(E_{\lambda_i}) \leq m_i$)

Remark: "Diagonalizable" (as a linear transf) $\iff$ eigenspaces are BIG enough

Theorem: Any $A \in M_{n \times n}(\mathbb{C})$ is similar to a matrix of the following form:

$$J = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & & 0 \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} \lambda_2 & 1 & & 0 \\ & \lambda_2 & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} \lambda_N & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_N \end{matrix}} \end{pmatrix}$$

(Jordan Canonical Form of A)

$\lambda_1, \lambda_2, \ldots, \lambda_N$ are eigenvalues of A (not necessarily distinct)

$$\begin{pmatrix} \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} & & \\ & \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} & \\ & & \begin{pmatrix} 3 & 1 \\ & 3 \end{pmatrix} \end{pmatrix}$$

Given $T: V \to V$, V is fin-dim.

Find a basis β of V $\ni$ $[T]_\beta = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$

(A_i = block square matrix)

$A_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ where λ is an eigenvalue of T.

<u>Remark</u>:

- $[T]_\beta$ is called the <u>Jordan Canonical form</u> of T
- A_i is called a <u>Jordan block</u> corresponding to λ
- β is called the <u>Jordan canonical basis</u>.

Remark: Jordan canonical consists of blocks in this form:

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \lambda \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in M_{k\times k}(\mathbb{C})$$

It is called Jordan block of size k with eigenvalue λ.

Prop: (1) A has only 1 eigenvalue λ (multiplicity is k)

(2) $\dim(E_\lambda) = 1$ ($\Rightarrow$ A is not diagonalizable if $k \neq 1$)

(3). The smallest positive integer p s.t.

$(A - \lambda I)^p = 0$ is equal to the dimension k.

($\Rightarrow$ $N((A - \lambda I)^p) = \mathbb{C}^k$)

(4) If $\{\vec{e}_1, \dots, \vec{e}_k\}$ is the standard basis for $\mathbb{C}^k$,

then $(A - \lambda I)^i \vec{e}_i = 0$ for each $i = 1, 2, \dots, k$.

$$K = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \Rightarrow \begin{cases} ① \ 1 \text{ eigenvalue} \\ ② \ \dim(E_\lambda) = 1 \end{cases}$$

$$N(K - \lambda I) = n - \underset{\substack{\parallel \\ R\left(\begin{pmatrix} 0 & 1 & 0 & & 0 \\ \vdots & 0 & 1 & \ddots & \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & 0 & 0 & & 0 \end{pmatrix}\right) = n-1}}{R(K - \lambda I)} = 1$$

If JCF of A is

$$J = \begin{pmatrix} \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} & & \\ & \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} & \\ & & \begin{pmatrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{pmatrix} \end{pmatrix}.$$

Then: $E_2 = N(J - 2I)$

$= 2$

$\neq$ algebraic multiplicity

$= 5$

$\therefore$ Not diagonalizable.

e.g. $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$. Then: $(A - \lambda I) = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$

$$(A - \lambda I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

$$(A - \lambda I)^3 = 0$$

<u>Definition:</u> Let λ be an eigenvalue of $A \in M_{n \times n}(\mathbb{C})$. $\vec{x} \in \mathbb{C}^n$ is a generalized eigenvector of A corresponding to the eigenvalue λ if (i) $\vec{x} \neq \vec{0}$

and (ii) $(A - \lambda I)^p \vec{x} = 0$ for some positive integer p.

We denote the generalized eigenspace by:

$$K_\lambda = \{ \vec{x} \in \mathbb{C}^n : (A - \lambda I)^p \vec{x} = \vec{0} \text{ for some } p \geq 1 \}$$

Main Theorem: (Jordan Decomposition Theorem)

Let $A \in M_{n\times n}(\mathbb{C})$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (distinct) with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Then:

(1) $\dim(K_{\lambda_i}) = m_i$

(2) $\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$

(3) Each K_{λ_i} has a basis $\beta_i = \gamma_{1,i} \cup \cdots \cup \gamma_{\ell,i}$ where every $\gamma_{m,i}$ is a cycle:

$$\gamma_{m,i} = \left\{ (A-\lambda_i I)^{p-1}\vec{x}, (A-\lambda_i I)^{p-2}\vec{x}, \quad \ldots, (A-\lambda_i I)\vec{x}, \vec{x} \right\}$$

Consider $\gamma = \{ \underbrace{(A-\lambda_i I)^{p-1}\vec{x}}_{\vec{w}_1},\ \underbrace{(A-\lambda_i I)^{p-2}\vec{x}}_{\vec{w}_2},\ \ldots,\ \underbrace{\vec{x}}_{\vec{w}_p} \}$

$$\vec{w}_1 = (A-\lambda_i I)\vec{w}_2 \Rightarrow A\vec{w}_2 = \vec{w}_1 + \lambda_i \vec{w}_2$$

$$[L_A]_\gamma = \begin{pmatrix} & | & | & \\ & [A\vec{w}_1]_\gamma & [A\vec{w}_2]_\gamma & \\ & | & | & \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_i & 1 & 0 & & \\ 0 & \lambda_i & 1 & & 0 \\ 0 & 0 & \lambda_i & \ddots & \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & \cdots & \cdots & & \lambda_i \end{pmatrix}$$

$\Rightarrow$ Jordan block.

Example: $A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$

Step 1: Compute eigenvalues.

$f(t) = -(t-1)^3 \Rightarrow$ ONLY 1 eigenvalue $\lambda = 1$, alg. mult $= 3$.

Step 2: Find eigenspace

$$E_1 = N(A - 1I) = N\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$\dim(E_1) = 2 < 3$ ($\Rightarrow A$ is NOT diagonalizable)

Step 3: Decide the Jordan Canonical Form

$$\cancel{J = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{matrix}} \end{pmatrix}} \quad \text{or} \quad \cancel{J = \begin{pmatrix} \boxed{1} & & \\ & \boxed{1} & \\ & & \boxed{1} \end{pmatrix}} \quad \text{or} \quad J = \begin{pmatrix} \boxed{1} & & \\ & \boxed{\begin{matrix} 1 & 1 \\ & 1 \end{matrix}} \end{pmatrix}$$

Step 4: Find basis K_λ consisting of cycles.

$$\beta = \gamma_1 \cup \gamma_2 = \{\vec{v}_1\} \cup \underbrace{\{(A - \underset{1}{\underset{\shortparallel}{\lambda}} I)\vec{v}_2, \vec{v}_2\}}_{\gamma_2}$$

eigenvectors

Need $\vec{v}_2 \in N\left((A-\lambda I)^2\right)$ but $\vec{v}_2 \notin N(A-\lambda I) = E_1$

$$(A - 1I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Then: $\gamma_2 = \{(A-\lambda I)\vec{v}_2, \vec{v}_2\} = \left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

Take $\vec{v}_1 \in E_1 \Rightarrow \vec{v}_1$ is not parallel to $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Choose $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad Q^{-1} A Q = \begin{pmatrix} \boxed{1} & \\ & \boxed{\begin{matrix} 1 & 1 \\ & 1 \end{matrix}} \end{pmatrix} \quad \text{where} \quad Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

Example: $A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$.

Step 1: Eigenvalues $\lambda = -1$, alg. mult. $= 3$

Step 2: Eigenspace $E_{-1} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow \dim(E_{-1}) = 1 < 3$

Step 3: Jordan Canonical Form

$$J = \begin{pmatrix} -1 & 1 & 0 \\ & -1 & 1 \\ 0 & & -1 \end{pmatrix}$$

Step 4: Find the basis for K_{-1}

$$\beta = \{ (A+I)^2 \vec{v},\ (A+I)\vec{v},\ \vec{v} \}$$

Find $\vec{v} \in N((A+I)^3)$ but $\vec{v} \notin N((A+I))$

$\vec{v} \notin N((A+I)^2)$

$$N(A+I) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$$

$$N((A+I)^2) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

Take $\vec{v} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Then: $\beta = \left\{\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

$$\Rightarrow Q^{-1}AQ = \begin{pmatrix}-1 & 1 & 0\\0 & -1 & 1\\0 & 0 & -1\end{pmatrix} \text{ where } Q = \begin{pmatrix}2 & 0 & 0\\0 & -2 & 0\\0 & 0 & 1\end{pmatrix}$$

Theoretical proof: $T: V \to V$ (fin-dim), char poly splits.
($\lambda_1, \lambda_2, \ldots, \lambda_k$ distinct eigenvalues)

Claim 1: $(T - \lambda_i I)\big|_{K_{\lambda_j}} : K_{\lambda_j} \to K_{\lambda_j}$ is 1-1 if $i \neq j$.

Pf: Let $\vec{x} \in K_{\lambda_j}$ and $(T - \lambda_i I)(\vec{x}) = \vec{0}$.

Let $p =$ smallest integer $\ni$ $(T - \lambda_j I)^p(\vec{x}) = \vec{0}$.

Let $\vec{y} = (T - \lambda_j I)^{p-1}(\vec{x}) \neq \vec{0}$. Then: $(T - \lambda_j I)(\vec{y}) = \vec{0}$

$\therefore \quad \vec{y} \in E_{\lambda_j}$

Also, $(T - \lambda_i I)(\vec{y}) = (T - \lambda_i I)(T - \lambda_j I)^{p-1}(\vec{x})$

$$= (T - \lambda_j I)^{p-1} \underbrace{(T - \lambda_i I)(\vec{x})}_{\vec{0}} = \vec{0}$$

$\therefore \quad \vec{y} \in E_{\lambda_i}$

$\Rightarrow \vec{y} \in E_{\lambda_i} \cap E_{\lambda_j} = \{\vec{0}\} \Rightarrow \vec{y} = \vec{0}$ (Contradiction)

$N((T - \lambda_j I|_{K_{\lambda_j}}) = \{\vec{0}\} \Rightarrow (T - \lambda_j I)\big|_{K_{\lambda_j}}$ is 1-1.

<u>Claim 2:</u> $\dim(K_{\lambda_i}) \leq m_i$ = multiplicity of λ_i and

$$K_{\lambda_i} = N\left((T - \lambda_i I)^{m_i}\right)$$

<u>Pf:</u>

1 Let $g(t)$ = char poly of $T|_{K_{\lambda_i}}$

Then $g(t) \mid$ char poly of T.

Now, $(T - \lambda_j I)|_{K_{\lambda_i}}(\vec{x}) \neq \vec{0}$ if $\lambda_i \neq \lambda_j$ and $\vec{x} \neq \vec{0}$

$\therefore$ λ_i is the ONLY eigenvalue of $T|_{K_{\lambda_i}}$

$\therefore$ $g(t) = (\lambda_i - t)^d$, $d = \dim(K_{\lambda_i})$

$\therefore$ $d \leq m_i$

② $N((T-\lambda_i I)^{m_i}) \subseteq K_{\lambda_i}$ (by definition)

Now, by Caley - Hamilton Thm,

$$g(T|_{K_{\lambda_i}}) = 0$$

Char poly of $T|_{K_{\lambda_i}}$

$\therefore$ $(T|_{K_{\lambda_i}} - \lambda_i I)^d = 0 \Rightarrow (T-\lambda_i I)^d(\vec{x}) = \vec{0}$ for $\forall \vec{x} \in K_{\lambda_i}$

$(d \leq m_i)$

$\Rightarrow (T-\lambda_i I)^{m_i}(\vec{x}) = \vec{0}$ for $\forall \vec{x} \in K_{\lambda_i}$

$$\therefore K_{\lambda_i} \subseteq N((T-\lambda_i I)^{m_i})$$

Claim 3: $V = K_{\lambda_1} + K_{\lambda_2} + \dots + K_{\lambda_k}$

Pf: By M.I. on k = # of distinct eigenvalues.

When $k=1$, let m = multiplicity of λ_1. Then, char poly of T " $(\lambda_1 - t)^m$.

By Caley-Hamilton Thm, $g(T) = (\lambda_1 I - T)^m = 0 \leftarrow$ zero transf.

$$\therefore \quad K_{\lambda_1} = N((T - \lambda_1 I)^m) = V$$

$\therefore$ Thm is true for $k=1$.

Assume that the thm is true for any lin. op. w/ fewer than k distinct eigenvalues.

Consider $T: V \to V$ with k distinct eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_k$