Lecture 18:

Recall:

__Theorem:__ Let T be a linear operator on a finite-dim complex inner product space V. Then, T is normal iff $\exists$ an orthonormal basis for V consisting of eigenvectors of T.

Example: Let H be the set of continuous complex-valued functions defined on $[0, 2\pi]$ equipped w/ the inner product

$$\langle f, g \rangle \overset{\text{def}}{:=} \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\overline{g(t)}\,dt \quad \text{for } f, g \in H.$$

and the orthonormal subset:

$$S = \{ f_n(t) \overset{\text{def}}{:=} e^{int} : n \in \mathbb{Z} \} \subset H$$

Let $\overset{\text{inf dim}}{\overset{\shortparallel}{V}} = \text{span}(S)$ and consider the operators T and U on V defined by:

$$T(f) = f_1 \cdot f = e^{it} f, \qquad U(f) = f_{-1} f \overset{\shortparallel}{} e^{-it} f$$

$\therefore \quad T(f_n) = f_{n+1} \quad \text{and} \quad U(f_n) = f_{n-1} \quad \forall n \in \mathbb{Z}.$

$$e^{it} e^{int} = e^{i(n+1)t}$$

Then:
$$\langle T(f_m), f_n \rangle = \langle f_{m+1}, f_n \rangle = \delta_{m+1,n} = \begin{cases} 1 & \text{if } m+1=n \\ 0 & \text{otherwise} \end{cases}$$
$$= \delta_{m,n-1}$$
$$= \langle f_m, f_{n-1} \rangle$$
$$= \langle f_m, U(f_n) \rangle$$

$$\Rightarrow \quad U = T^*$$

$$\therefore \quad TT^* = TU = I = T^*T = UT. \quad \therefore \ T \text{ is normal.}$$

However, T has no eigenvectors.

If $f \in V$ is an eigenvector of T, say, $T(f) = \lambda f$ $(\lambda \in \mathbb{C})$

Then, we write $f = \sum_{i=n}^{m} a_i f_i$, where $a_m \neq 0$

$$\therefore \sum_{i=n}^{m} a_i f_{i+1} = T(f) = \lambda f = \sum_{i=n}^{m} \lambda a_i f_i$$

$$\Rightarrow f_{m+1} = \frac{1}{a_m}\left(\lambda a_n f_n + \sum_{i=n+1}^{m} (\lambda a_i - a_{i-1}) f_i\right)$$

Contradicting the fact that S is linearly independent.

Def: Let T be a linear operator on an inner product space V. We say T is self-adjoint (Hermitian) if $T^* = T$.

An $n \times n$ real or complex matrix A is called self-adjoint (or Hermitian) if $A^* = A$.

Lemma: Let T be a self-adjoint linear operator on a fin-dim inner product space V. Then:

(a) Every eigenvalue of T is real.

(b) Suppose V is real inner product space. Then, the char. poly of T splits over $\mathbb{R}$.

Proof: (a) Suppose $T(\vec{x}) = \lambda\vec{x}$ for $\vec{x} \neq \vec{0}$.

Then: $T^*(\vec{x}) = \bar{\lambda}\vec{x}$ ($\because$ T is normal)

$$\therefore \lambda\vec{x} = T(\vec{x}) = T^*(\vec{x}) = \bar{\lambda}\vec{x}$$

$$\therefore (\lambda - \bar{\lambda})\underbrace{\vec{x}}_{\neq \vec{0}} = \vec{0} \Rightarrow \lambda = \bar{\lambda}. \therefore \lambda \text{ is real.}$$

(b) Let $n = \dim(V)$, β be an orthonormal basis for V and let $A \overset{\text{def}}{=} [T]_\beta$

Then: A is self-adjoint. Consider: $L_A : \mathbb{C}^n \to \mathbb{C}^n$

By (a), the eigenvalues of L_A are real.

By Fundamental Thm of Algebra, $f_{L_A}(t)$ splits into factors of the form $t - \lambda$ where λ is an eigenvalue of L_A.

$\because$ λ is real $\therefore$ $f_{L_A}(t)$ splits over $\mathbb{R}$.

But $f_T(t) = f_{L_A}(t)$. So, the result follows.

Theorem: Let T be a linear operator on a fin-dim real inner product space V. Then T is self-adjoint iff $\exists$ orthonormal basis for V consisting of eigenvectors of T.

Proof: ($\Rightarrow$) Suppose T is self-adjoint. By the Lemma, the char poly of T splits over $\mathbb{R}$. By Schur's Theorem, $\exists$ an orthonormal basis β for V s.t. $A \overset{\text{def}}{=} [T]_\beta$ is upper triangular. But:

$$A^* = ([T]_\beta)^* = [T^*]_\beta = [T]_\beta = A$$

So, A is both upper triangular and lower triangular.
Hence, A is diagonal.
$\therefore$ β consists of eigenvectors of T.

($\Leftarrow$) Suppose $\exists$ orthurnormal basis β for V s.t. $A = [T]_\beta$ is diagonal.

Then: $[T^*]_\beta = ([T]_\beta)^* = A^t = A = [T]_\beta$

$\therefore \quad T^* = T$

$\therefore \quad T$ is self-adjoint.

Def: Let T be a linear operator on finite-dim inner product space V over F. If $\|T(\vec{x})\| = \|\vec{x}\|$ $\forall \vec{x} \in V$, then we call T is a unitary linear operator. (resp. orthogonal operator) if $F = \mathbb{C}$ (resp $F = \mathbb{R}$)

Lemma: Let U be a self-adjoint linear operator on a fin-dim inner product space V. If $\langle \vec{x}, U(\vec{x}) \rangle = 0$ $\forall \vec{x} \in V$, then $U = T_0$ = zero transf.

Pf: Choose an orthonormal basis β for V consisting of eigenvectors of U.

If $\vec{x} \in \beta$, then $U(\vec{x}) = \lambda \vec{x}$ for some λ.

$$0 = \langle \vec{x}, U(\vec{x}) \rangle = \langle \vec{x}, \lambda \vec{x} \rangle = \bar{\lambda} \langle \vec{x}, \vec{x} \rangle = \bar{\lambda} \|\vec{x}\|^2$$

$$\Rightarrow \quad \lambda = 0$$

$\therefore \quad U(\vec{x}) = 0$ for $\forall \vec{x} \in \beta$

$\therefore \quad U = T_0$.

<u>Thm:</u> For a linear operator T on a fin-dim inner product space V, the following are equivalent:

(a) $TT^* = T^*T = I$

(b) T preserves the inner product on V, i.e.,

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle \quad \forall \vec{x}, \vec{y} \in V.$$

(c) $T(\beta) \overset{\text{def}}{=} \{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ is an orthonormal basis for V for any orthonormal basis β for V

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

(d) $\exists$ an orthonormal basis β for V s.t. $T(\beta)$ is an orthonormal basis for V.

(e) $\| T(\vec{x}) \| = \| \vec{x} \|$ for $\forall \vec{x} \in V$

Proof: (a) $\Rightarrow$ (b) : $\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \underbrace{T^* T}_{= I}(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$

(b) $\Rightarrow$ (c) : Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be an orthonormal basis for V

Then $\langle T(\vec{v}_i), T(\vec{v}_j) \rangle = \langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore$ $T(\beta)$ is an orthonormal basis for V.

(c) $\Rightarrow$ (d) : Obvious

(d) $\Rightarrow$ (e) : Let $\vec{x} \in V$, and $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ = o.n. basis for V.

$\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i$ for some $a_1, \ldots, a_n \in F$. $\Rightarrow \|\vec{x}\|^2 \overset{\text{def}}{=} \langle \vec{x}, \vec{x} \rangle = \sum_{i=1}^{n} |a_i|^2$

(Check)

But $T(\beta)$ is o.n. basis.

$\therefore \|T(\vec{x})\|^2 = \left\| \sum_{i=1}^{n} a_i T(\vec{v}_i) \right\|^2 = \sum_{i=1}^{n} |a_i|^2$ $\quad\therefore \|T(\vec{x})\| = \|\vec{x}\|$ for $\forall \vec{x} \in V$.

$(e) \Rightarrow (a)$: $\forall \vec{x} \in V$, $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2 = \|T(\vec{x})\|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle$

$= \langle \vec{x}, T^* T(\vec{x}) \rangle$

$$\Rightarrow \langle \vec{x}, \overbrace{(I - T^* T)}^{U}(\vec{x}) \rangle = 0 \quad \text{for all } \vec{x} \in V.$$

$= $ Self-adjoint.

By lemma, we know $I - T^* T = T_0 \Rightarrow T^* T = I$

Similarly, we can show $T T^* = I$

Def: A matrix $A \in M_{n\times n}(\mathbb{R})$ is called orthogonal if:

$$A^T A = AA^T = I$$

The set of orthogonal real matrices is denoted as $O(n)$

A matrix $A \in M_{n\times n}(\mathbb{C})$ is called unitary if:

$$A^* A = AA^* = I$$

The set of unitary complex matrices is denoted as $U(n)$

Remark:

- T is unitary (or orthogonal) iff $\exists$ an o.n. basis β s.t. $[T]_\beta$ is unitary (resp. orthogonal).

  $([T^*]_\beta = ([T]_\beta)^*)$

- Let $\vec{v}_1, \ldots, \vec{v}_n \in F^n$. Then: $A \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}(F)$ is unitary (or orthogonal) iff $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is an o.n. basis for $\mathbb{C}^n$ (resp $\mathbb{R}^n$)

Thm: Let $A \in M_{n\times n}(\mathbb{C})$. Then: L_A is normal iff A is unitarily equivalent to a diagonal matrix.

(That is, $\exists\, P \in U(n)$ s.t. P^*AP is diagonal)

Pf: ($\Rightarrow$) Suppose L_A is normal. Then: $\exists$ an o.n. basis β, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of eigenvectors for $\mathbb{C}^n$ s.t. $[L_A]_\beta = P^{-1} \underset{[L_A]_{\beta'}}{\underset{\shortparallel}{A}} P$

β' = standard ordered basis

is diagonal, where $P = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$

$\because$ P is unitary, $P^*P = PP^* = I \Rightarrow P^{-1} = P^*$.

($\Leftarrow$) Obvious. Exercise.

Thm: Let $A \in M_{n \times n}(\mathbb{R})$. Then: A is symmetric iff A is orthogonally equivalent to a diagonal matrix.

That is, $\exists P \in O(n)$ s.t. $P^T A P$ is diagonal.

e.g.

Consider $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$. Then $\exists P \in O(3)$ s.t. $P^t A P$ is diagonal.

To find P explicitly, we first compute the eigenvalues of A:

$$f_A(t) = (8-t)(2-t)^2$$

So the eigenvalues are $\lambda = 2$ and $\lambda = 8$

For $\lambda = 8$, $(1,1,1)$ is an eigenvector

For $\lambda = 2$, $\{(-1,1,0), (-1,0,1)\}$ is a basis for the eigenspace E_2 but it is not orthogonal.

Applying the Gram-Schmidt process produces the orthogonal basis $\{(-1,1,0), (1,1,-2)\}$ of E_2.

Then an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A is given by

$$\left\{ \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(1,1,-2), \frac{1}{\sqrt{3}}(1,1,1) \right\}$$

which gives P as

$$P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2\sqrt{6} & 1/\sqrt{3} \end{pmatrix}.$$