Lecture 17:

Recall:

Let V be a finite-dim inner product space. Let T be a linear operator on V.

Then: $\exists!$ linear operator $T^* : V \to V$ such that:

$$\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle \text{ for } \forall \vec{x}, \vec{y} \in V.$$

T^* is called the adjoint of T.

Proposition: Let V be an inner product space. Let $T, U = V \to V$.

Then:

(a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \bar{c}\,T^* \quad \forall c \in F$

(c) $(TU)^* = U^* T^*$

(d) $(T^*)^* = T$

(e) $I^* = I$

Proof: $\forall \vec{x}, \vec{y} \in V$

(a) $\langle \vec{x}, (T+U)^*(\vec{y}) \rangle = \langle (T+U)(\vec{x}), \vec{y} \rangle = \langle T(\vec{x}), \vec{y} \rangle + \langle U(\vec{x}), \vec{y} \rangle$

$= \langle \vec{x}, T^*(\vec{y}) \rangle + \langle \vec{x}, U^*(\vec{y}) \rangle$

$= \langle \vec{x}, (T^* + U^*)(\vec{y}) \rangle$

$\Rightarrow (T+U)^* = T^* + U^*.$

(b) $$\langle \vec{x}, (cT)^*(\vec{y}) \rangle = \langle cT(\vec{x}), \vec{y} \rangle$$
$$= c \langle T(\vec{x}), \vec{y} \rangle$$
$$= c \langle \vec{x}, T^*(\vec{y}) \rangle = \langle \vec{x}, \bar{c}\, T^*(\vec{y}) \rangle$$

$$\therefore \quad (cT)^* = \bar{c}\, T^*$$

(c) $$\langle \vec{x}, (TU)^*(\vec{y}) \rangle = \langle T(U(\vec{x})), \vec{y} \rangle$$
$$= \langle U(\vec{x}), T^* \vec{y} \rangle$$
$$= \langle \vec{x}, U^* T^* \vec{y} \rangle$$

$$\Rightarrow \quad (TU)^* = U^* T^*.$$

(d) $\langle \vec{x}, T(\vec{y}) \rangle = \langle T^*(\vec{x}), \vec{y} \rangle = \langle \vec{x}, (T^*)^*(\vec{y}) \rangle$

$\Rightarrow \quad T = T^{**}$.

(e). follows from the definition.

$\langle \vec{x}, I(\vec{y}) \rangle = \langle I(\vec{x}), \vec{y} \rangle$

$\quad \parallel$

$\langle \vec{x}, \vec{y} \rangle$

<u>Remark:</u> Let A and B be $n \times n$ matrices. Then:

(a) $(A+B)^* = A^* + B^*$

(b) $(cA)^* = \bar{c} A^*$

(c) $(AB)^* = B^* A^*$

(d) $A^{**} = A$

(e) $I^* = I$.

Lemma: Let $T: V \to V$ be a linear operator on a finite-dim inner product space V. If T has an eigenvector, then so does T^*.

Pf: Suppose $\vec{v} \in V \setminus \{\vec{0}\}$ is an eigenvector of T with eigenvalue λ.

Then: $\forall \vec{x} \in V$, we have:

$$0 = \langle \vec{0}, \vec{x} \rangle = \langle (T - \lambda I)(\vec{v}), \vec{x} \rangle = \langle \vec{v}, \underbrace{\overset{T^* - \bar{\lambda} I \,=}{(T - \lambda I)^*}(\vec{x})}_{\in\, R(T^* - \bar{\lambda} I)} \rangle$$

$\Rightarrow \vec{v} \in R(T^* - \bar{\lambda} I)^{\perp}$. So, $\dim(R(T^* - \bar{\lambda} I)) < \dim(V)$.

$(\dim(W) + \dim(W^{\perp}) = \dim(V))$

$\Rightarrow \dim(N(T^* - \bar{\lambda} I)) > 0$ $\therefore$ T^* has an eigenvector with eigenvalue $\bar{\lambda}$.

Thm (Schur) Let T be a lin. operator on a finite-dim inner product space. Suppose the char. poly of T splits.

Then: $\exists$ an orthonormal basis β for V s.t. $[T]_\beta$ is upper triangular.

Pf: We prove by induction on $n = \dim(V)$.

The $n=1$ case is obvious.

Assume the statement holds for lin. operators defined on $(n-1)$-dim inner product space, whose char. poly splits.

By lemma, T^* has a unit eigenvector $\vec{z}$.

Let $W :\overset{\text{def}}{=} \operatorname{span}\{\vec{z}\}$ and suppose $T^*(\vec{z}) = \lambda \vec{z}$.

Claim: $W^\perp$ is T-invariant.

Pf: Let $\vec{y} \in W^\perp$ and $\vec{x} = c\vec{z} \in W$. Then:

$$\langle T(\vec{y}), \vec{x} \rangle = \langle T(\vec{y}), c\vec{z} \rangle = \langle \vec{y}, c\, T^*(\vec{z}) \rangle$$
$$= \langle \vec{y}, c\lambda \vec{z} \rangle$$
$$= \bar{c}\bar{\lambda} \langle \underset{W^\perp}{\vec{y}}, \underset{W}{\vec{z}} \rangle = 0$$

$\therefore$ $T(\vec{y}) \in W^\perp$.

Now, $f_{T_{W^\perp}}(t) \mid f_T(t) \Rightarrow f_{T_{W^\perp}}(t)$ splits. ①

Also, $\dim(W^\perp) = n-1$ ②

$\therefore$ Induction hypothesis gives an orthonormal basis γ for $W^\perp$ s.t. $[T_{W^\perp}]_\gamma$ is upper triangular.

Then: $\beta \overset{def}{=} \underset{\substack{\cap \\ W^{\perp}}}{\gamma} \cup \{ \underset{\substack{\cap \\ W}}{\vec{z}} \}$ is orthonormal basis s.t.

$$[T]_\beta = \begin{pmatrix} [T_{W^\perp}]_\gamma & \begin{matrix} * \\ * \\ * \\ * \\ \vdots \end{matrix} \\ 0 \cdots\cdots 0 & * \end{pmatrix}$$ is upper triangular

Assume T is diagonalizable and assume $\exists$ an orthonormal basis β for V s.t. $[T]_\beta$ is diagonal.

Then: $[T^*]_\beta = ([T]_\beta)^*$ is also diagonal

$$\therefore \quad ([T]_\beta)^* ([T]_\beta) = ([T]_\beta)([T]_\beta)^*$$

$$\begin{aligned} &\parallel \\ &[T^*]_\beta [T]_\beta = [T]_\beta [T^*]_\beta \\ &\parallel \\ &[T^*T]_\beta = [TT^*]_\beta \end{aligned}$$

$$\Rightarrow \quad T^*T = TT^*$$

Definition: Let V be an inner product space. We say that a linear operator $T: V \to V$ is **normal** if $T^*T = TT^*$.

An $n \times n$ real or complex matrix A is called **normal** if

$$A^* A = A A^*$$

Example:

- Unitary (when $F = \mathbb{C}$) or orthogonal (when $F = \mathbb{R}$) if $T^*T = TT^* = I$
- Hermitian (or self-adjoint) if $T^* = T$
- Skew-Hermitian (or anti-self-adjoint) if $T^* = -T$.

Are normal!

Proposition: Let V be an inner product space, and let T be a normal linear operator on V. Then: we have:

(a) $\|T(\vec{x})\| = \|T^*(\vec{x})\| \quad \forall x \in V$

(b) $T - cI$ is normal $\forall c \in F$.

(c) If $T(\vec{x}) = \lambda\vec{x}$, then: $T^*(\vec{x}) = \overline{\lambda}\vec{x}$

(d) If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$, then: $\vec{x}_1$ and $\vec{x}_2$ are orthogonal.

Proof: (a) $\forall \vec{x} \in V$, we have:

$$\|T(\vec{x})\|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle = \langle T^* T(\vec{x}), \vec{x} \rangle$$

$$= \langle T T^*(\vec{x}), \vec{x} \rangle = \langle T^*(\vec{x}), T^*(\vec{x}) \rangle$$

$$= \|T^*(\vec{x})\|^2$$

(b). $$(T - cI)(T - cI)^* = (T - cI)(T^* - \bar{c} I)$$

$$= TT^* - cT^* - \bar{c}T + c\bar{c}I$$

$$= T^*T - cT^* - \bar{c}T + c\bar{c}I$$

$$= (T - cI)^*(T - cI).$$

(c) Suppose $T(\vec{x}) = \lambda \vec{x}$. Let $U = T - \lambda I$. Then, U is normal (by (b)) and $U(\vec{x}) = \vec{0}$. So, by (a),

$$0 = \|U(\vec{x})\| = \|U^*(\vec{x})\| = \|(T^* - \bar{\lambda} I)(\vec{x})\| \iff T^*(\vec{x}) = \bar{\lambda} \vec{x}.$$

(d) By (c), we have:

$$\lambda_1 \langle \vec{x}_1, \vec{x}_2 \rangle = \langle T(\vec{x}_1), \vec{x}_2 \rangle = \langle \vec{x}_1, T^*(\vec{x}_2) \rangle$$

$$= \langle \vec{x}_1, \bar{\lambda}_2 \vec{x}_2 \rangle$$

$$= \lambda_2 \langle \vec{x}_1, \vec{x}_2 \rangle$$

$\lambda_1 \neq \lambda_2$

$$\Longleftrightarrow \quad \overset{\neq 0}{(\lambda_1 - \lambda_2)} \langle \vec{x}_1, \vec{x}_2 \rangle = 0$$

$$\Longrightarrow \quad \langle \vec{x}_1, \vec{x}_2 \rangle = 0$$

Theorem: Let T be a linear operator on a finite-dim complex inner product space V. Then, T is normal iff $\exists$ an orthonormal basis for V consisting of eigenvectors of T.

Proof: $(\Leftarrow)$ Obvious.

$(\Rightarrow)$ Suppose T is normal.

By the Fundamental Thm of algebra, $f_T(t)$ splits.

$\therefore$ Schur's Theorem gives us an orthonormal basis $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ s.t. $[T]_\beta$ is upper triangular.

$$[T]_\beta = \begin{pmatrix} [T(\vec{v}_1)]_\beta & \cdots \end{pmatrix}.$$

In particular, $\vec{v}_1$ is an eigenvector of T.

Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}$ are eigenvectors of T and $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$ are their corresponding eigenvalues

We claim that $\vec{v}_k$ is an eigenvector of T (so by induction, all vectors in β are eigenvectors of T)

Now, $T(\vec{v}_j) = \lambda_j \vec{v}_j \Rightarrow T^*(\vec{v}_j) = \bar{\lambda}_j \vec{v}_j$ for $j = 1, 2, \dots, k-1$

$\because$ $A \overset{\text{def}}{:=} [T]_\beta$ is upper triangular

$$T(\vec{v}_k) = A_{1k}\vec{v}_1 + A_{2k}\vec{v}_2 + \dots + A_{kk}\vec{v}_k$$

But : $A_{jk} = \langle T(\vec{v}_k), \vec{v}_j \rangle = \langle \vec{v}_k, T^*(\vec{v}_j) \rangle = \langle \vec{v}_k, \bar{\lambda}_j \vec{v}_j \rangle$

$$= \lambda_j \underbrace{\langle \vec{v}_k, \vec{v}_j \rangle}_{= 0}$$

$$= 0$$

for $j = 1, 2, \dots, k-1$. $\therefore$ $T(\vec{v}_k) = A_{kk}\vec{v}_k$

$\therefore$ $\vec{v}_k$ = eigenvector of T.