Lecture 13:

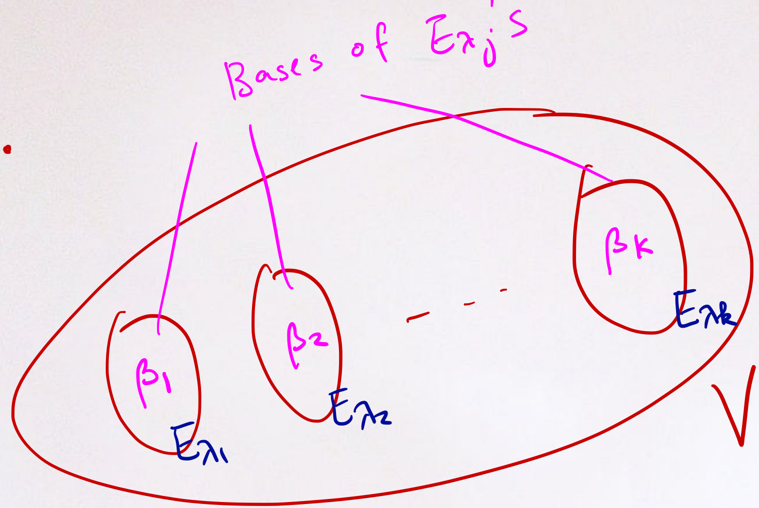
Recall: $T : V \to V$, $F = \mathbb{C}$.

Char poly of T : $f_T(t) = (-1)^n (t-\lambda_1)^{n_1} (t-\lambda_2)^{n_2} \cdots (t-\lambda_k)^{n_k}$

n_j = algebraic multiplicity of λ_j = $\mu_T(\lambda_j)$

$$\underbrace{\dim\left(\text{Eigenspace of } \lambda_j\right) = \dim\left(N(T - \lambda_j I_V)\right) = \dim\left(E_{\lambda_j}\right)}_{\text{Geometric multiplicity } = \gamma_T(\lambda_j)}$$

- T is diagonalizable iff $\mu_T(\lambda_j) = \gamma_T(\lambda_j)$ for $j = 1, 2, \ldots, k$

- 

Then: $\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$ is a basis of eigenvectors for V.

Example: Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by:

$$T(f(x)) = f(x) + (x+1)f'(x)$$

Then: $A := [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

where $\beta = \{1, x, x^2\}$ = standard ordered basis for $P_2(\mathbb{R})$.

$\therefore$ the char. poly:

$$\det(A - tI_3) = \det\begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} = (1-t)^1(2-t)^1(3-t)^1$$

$$\left.\begin{aligned} 1 &\le \gamma_T(1) \le \mu_T(1) = 1 \\ 1 &\le \gamma_T(2) \le \mu_T(2) = 1 \\ 1 &\le \gamma_T(3) \le \mu_T(3) = 1 \end{aligned}\right\} \quad \begin{aligned} \gamma_T(1) &= \mu_T(1) \\ \gamma_T(2) &= \mu_T(2) \\ \gamma_T(3) &= \mu_T(3) \end{aligned} \Rightarrow \text{Diagonalizable}$$

$$\cancel{E_1 =}\ N(\underbrace{\overset{=[T]_\beta}{A} - 1 I_3}_{[T - 1 I_V]_\beta}) = N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \left\{ a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

$$\Rightarrow \quad E_1 = N(T - 1 I_V) = \{ a 1 : a \in \mathbb{R} \} \subseteq P_2(\mathbb{R})$$

Similarly, $N(A - 2I_3) = N\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \left\{ a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}$

$$E_2 = \{ a(1+x) : a \in \mathbb{R} \} \subset P_2(\mathbb{R})$$

$$E_3 = \{ a \overbrace{(1 + 2x + x^2)}^{(1+x)^2} : a \in \mathbb{R} \}$$

$\beta = \{1, 1+x, (1+x)^2\}$ is a basis of eigenvectors for V.

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "{[T]_\beta}"] \arrow[d, leftrightarrow] & \mathbb{R}^3 \arrow[d, leftrightarrow] \\ P_2(\mathbb{R}) \arrow[r, "T"] & P_2(\mathbb{R}) \end{tikzcd}$$

Example: For $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$

$f_A(t) = -(t-4)(t-3)^2$ splits over $\mathbb{R}$.

$\gamma_T(4) = \mu_T(4) = 1$

But $\operatorname{rank}(\overbrace{A - 3I}^{B}) = \operatorname{rank}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2$

$(\underbrace{\operatorname{Rank}(B)}_{2} + \underbrace{\operatorname{Nullity}(B)}_{1} = 3)$

$\gamma_A(3) = 1 \neq \mu_A(3) = 2$

$\therefore$ T is not diagonalizable.

Example: Consider $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by:

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$$

Let $\beta = \{1, x, x^2\}$.

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow f_T(t) = -(t-1)^2(t-2)$$

splits over $\mathbb{R}$.

and the eigenvalues of T are 1 and 2.

$\therefore \gamma_T(2) = \mu_T(2) = 1$.

$$\operatorname{Rank}([T]_\beta - I) = \operatorname{rank}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1 \Rightarrow \gamma_T(1) = 2 = \mu_T(1)$$

$\therefore T$ is diagonalizable.

For $[T]_\beta$, the eigenspaces:

$$E_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 + x_3 = 0 \right\} = \operatorname{span} \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}}_{\text{Basis}}$$

$$E_2 = \operatorname{span} \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{Basis}} \right\}$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of eigenvectors (of $[T]_\beta$) for $\mathbb{R}^3$.

$\therefore \{1, x - x^2, 1 + x^2\}$ is a basis of eigenvectors (of T) for $P_2(\mathbb{R})$.

Definition: Let T be a linear operator on a vector space V.

A subspace $W \subset V$ is called **T-invariant** if $T(W) \subseteq W$.

That is, $T(\vec{w}) \in W$ for $\forall \vec{w} \in W$.

Example: If T is a linear operator on V, then:

$\{\vec{0}\}$ is T-invariant

V is " "

$R(T)$ " "

$N(T)$ " "

E_λ " "

$\uparrow$ eigenvalue

$\Big(\underset{T(\vec{v})}{\vec{w}} \in R(T)$, then: $\underset{\in R(T)}{T(T(\vec{v}))} \Big)$

$\Big(\vec{v} \in E_\lambda,\ T(\vec{v}) = \underset{\in E_\lambda}{\lambda \vec{v}} \Big)$

- For $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a,b,c) = (a+b,\ b+c,\ 0)$

then x-y plane $\{(x, y, 0) : x, y \in \mathbb{R}\}$ is T-invariant

x-axis $\{(x, 0, 0) : x \in \mathbb{R}\}$ is T-invariant

z-axis $\{(0, 0, x) : x \in \mathbb{R}\}$ is NOT T-invariant.

$T((0, 0, \underset{\neq 0}{x})) = (0, \underset{\neq 0}{x}, 0) \notin$ z-axis

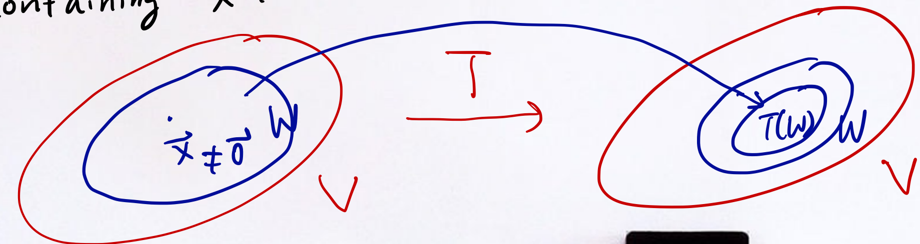
Def: Given a linear operator T on a vector space V, and a non-zero $\vec{x} \in V$, the subspace

$$W := \operatorname{span}\left(\{ T^k(\vec{x}) : k \in \mathbb{N} \}\right) \overset{\text{def}}{=} \operatorname{span}\left(\{ \vec{x}, T(\vec{x}), T^2(\vec{x}), \ldots, T^k(\vec{x}), \ldots \}\right)$$

$$\left(T^k \overset{\text{def}}{=} \underbrace{T \circ T \circ \cdots \circ T}_{k \text{ times}} \right)$$

is called **T- cyclic subspace** of V generated by $\vec{x}$.

Prop: W is the smallest T-invariant subspace of V containing $\vec{x}$.

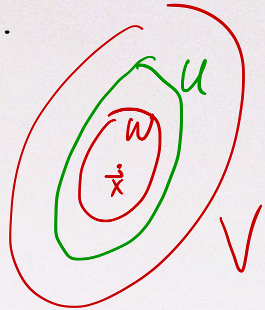


<u>Proof:</u> For any $\vec{w} \in W$, $\exists\, a_0, \dots, a_k \in F$ s.t.

$$\vec{w} = \sum_{i=0}^{k} a_i T^i(\vec{x})$$

Then: $T(\vec{w}) = \sum_{i=0}^{k} a_i T^{i+1}(\vec{x}) \in W$.

$\therefore$ W is T-invariant.



If $U \subset V$ is a T-invariant subspace containing $\vec{x}$.

then: it also contains $\underset{\in U}{T(\vec{x})}$ and $\underset{\in U}{T^k(\vec{x})}$ by induction.

$\therefore$ $U \supset W$

Example:

- For $T: P(\mathbb{R}) \to P(\mathbb{R})$ defined by $T(f(x)) = f'(x)$

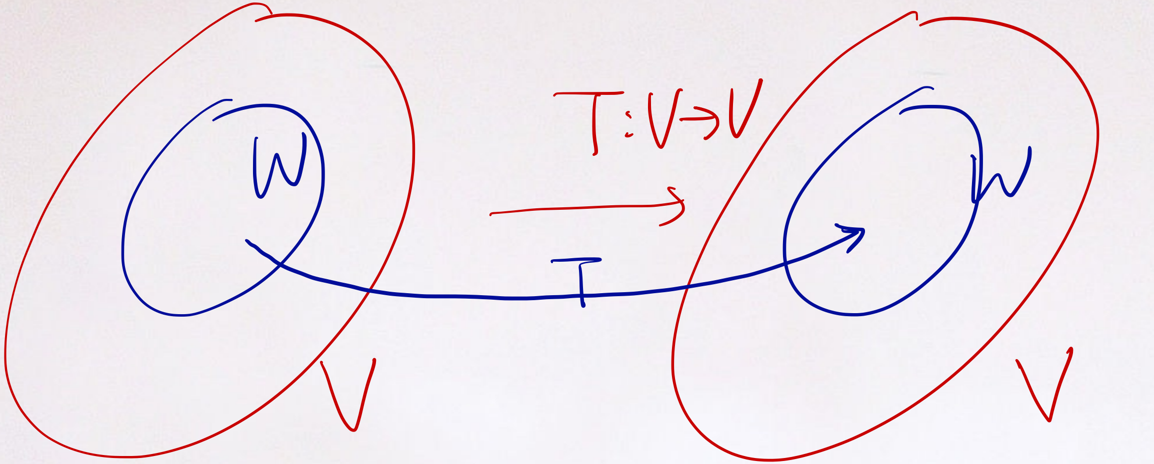
  then T-cyclic subspace generated by x^n is:

  $$\text{span}\left(\{x^n, nx^{n-1}, \ldots, n!\,x, n!\}\right) = P_n(\mathbb{R})$$

- Let $T: V \to V$ be linear. Then, a 1-dimensional T-invariant subspace $U \subset V$ is nothing but the span of an eigenvector of T.

[If U = 1-dim T-invariant subspace.
Then, $U = \text{span}\{\underset{\neq \vec{0}}{\vec{v}}\}$. Then: $T(\vec{v}) \in U$
$\Rightarrow T(\vec{v}) = \lambda \vec{v}$. $\therefore$ $\vec{v}$ = eigenvector of T.

Also, if $\vec{v} \in V$ is an eigenvector of T, then T-cyclic subspace generated by $\vec{v}$ is also $\text{span}\{\vec{v}\}$. $\left(= \{\vec{v}, \underset{\lambda\vec{v}}{\cancel{T(\vec{v})}}, \underset{\lambda^2\vec{v}}{\cancel{T^2(\vec{v})}} \ldots\}\right)$



Def: $T|_W : W \to W$ defined by $T|_W(\vec{w}) = T(\vec{w})$

$f_{T|_W}(t)$ divides $f_T(t)$