- Definition of vector spaces: Try to think about some examples of vector spaces
- Definition of subspaces: Try to think about some examples of subspaces, why is it important?
- What is the linear combination? What is a Spanning set? What is linearly independent? What is the intuitive meaning of linearly dependence? How to check linearly independence?
- What is the definition of basis? What is the meaning of dimension?
- What is the Replacement Theorem? What is the geometric picture of the theorem?
- Try to recall how we can compute RREF? How to compute inverse? How to find the solution set of a linear system? How to determine the dimension of the solution set? What is null-space? What is column space?

Lecture 1: Vector spaces

Field

Definition: A field is a set F along with two binary operations: $+$ (addition) and $\cdot$ (multiplication) such that:

- For $\forall\, x, y \in F$, $x + y = y + x$ and $x \cdot y = y \cdot x$
- For $\forall\, x, y, z \in F$, $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- For $\forall x, y, z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$
- $\exists !$ element $0 \in F \ni \forall\, x \in F$, $x + 0 = x$
- $\exists !$ element $1 \in F \ni \forall\, x \in F$, $x \cdot 1 = x$
- For $\forall\, x \in F$, $\exists$ an element $(-x) \in F \ni x + (-x) = 0$
- For $\forall\, x \in F$ (excluding $x = 0$), $\exists$ an element $x^{-1} \in F \ni x \cdot x^{-1} = 1$

Remark:
- We often write xy for $x \cdot y$
- If F is finite, we say it is a finite field

Examples of field

1. $F = \mathbb{R}$
2. $F = \mathbb{C}$

$\Big\}$ Most often considered in Math 2048.

3. $F = \{\text{Rational numbers}\} = \{ p/q : p, q \in \mathbb{Z} \}$
4. Finite field of order p (where p is a prime number)

Define $F_p = \{0, 1, 2, \ldots, p-1\}$ and $+ / \cdot$ are defined as:

$+$: for $\forall x, y \in F_p$, $x + y$ are performed modulo p.

That is, $x + y$ is the remainder of $(x+y)/p$

$\cdot$: for $\forall x, y, \in F_p$, $x \cdot y$ is the remainder of $x \cdot y / p$.

$F_2 = \{0, 1\}$ is the binary field (important for information theories)

Vector Space

Goal: Build an abstract space (space of objects) simulating $\mathbb{R}^n$ or $\mathbb{C}^n$ (with addition and multiplication/scaled)

Definition: A vector space over F is a set V equipped w/ two operations:

(addition) $+ : V \times V \to V$, $\quad (\underset{V}{\vec{x}}, \underset{V}{\vec{y}}) \mapsto \vec{x} + \vec{y} \in V$

(scalar multiplication) $\cdot : F \times V \to V$, $\quad (\underset{F}{a}, \underset{V}{\vec{x}}) \mapsto a\vec{x} \in V$

satisfying 8 properties:

$$\left.\begin{array}{ll}
(\text{VS1}): & \vec{x}+\vec{y}=\vec{y}+\vec{x} \quad \forall \vec{x},\vec{y}\in V \\
(\text{VS2}): & (\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z}) \quad \forall \vec{x},\vec{y},\vec{z}\in V \\
(\text{VS3}): & \exists\, \vec{0}\in V \text{ s.t. } \vec{x}+\vec{0}=\vec{x} \quad \forall \vec{x}\in V \\
(\text{VS4}): & \forall \vec{x}\in V,\ \exists\, \vec{y}\in V \text{ s.t. } \vec{x}+\vec{y}=\vec{0} \quad \text{(inverse)}
\end{array}\right\} +$$

$$\left.\begin{array}{ll}
(\text{VS5}): & \underset{\substack{\uparrow \\ F}}{1}\,\vec{x}=\vec{x} \quad \forall \vec{x}\in V \\
(\text{VS6}): & (\underset{\substack{\uparrow \\ F}}{a}\,\underset{\substack{\uparrow \\ F}}{b})\vec{x}=a(b\vec{x}) \quad \forall a,b\in F,\ \forall \vec{x}\in V
\end{array}\right\} \cdot$$

$$\left.\begin{array}{ll}
(\text{VS7}): & \underset{\substack{\uparrow \\ F}}{a}\,(\underset{\substack{\uparrow \\ V}}{\vec{x}}+\underset{\substack{\uparrow \\ V}}{\vec{y}})=a\vec{x}+a\vec{y} \quad \forall a\in F,\ \forall \vec{x},\vec{y}\in V \\
(\text{VS8}): & (a+b)\vec{x}=a\vec{x}+b\vec{x} \quad \forall a,b\in F,\ \forall \vec{x}\in V
\end{array}\right\} +/\cdot$$

<u>Remark:</u> an element in F is called <u>scalar</u>.
an element in V is called <u>vector</u>.

Examples of vector spaces

- $F^n = \{(x_1, x_2, \dots, x_n) : x_j \in F \text{ for } j = 1, 2, \dots, n\}$ w/

$$(x_1, x_2, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

$$a(x_1, \dots, x_n) = (ax_1, ax_2, \dots, ax_n)$$

- $M_{m \times n}(F) = \{ m \times n$ matrices w/ entries in $F\}$ w/ matrix addition and scalar multiplication

- $P(F) = \{$ polynomials w/ coefficients in $F\}$ w/ polynomial addition and scalar multiplication.

- $F^\infty = \{(x_1, x_2, \dots) : x_j \in F, j = 1, 2, \dots\}$ w/ component-wise addition and scalar multiplication.

- $\mathrm{Sym}_{n \times n}(F) = \{\, n \times n$ symmetric matrices A w/ entries in F : $A^T = A \,\}$

- Let S be any non-empty set.

  Then: $\mathcal{F}(S, F) = \{$ functions $f : S \to F \}$

  is a vector space over F under:

$$(\underset{\in\,\mathcal{F}(S,F)}{f} + \underset{\in\,\mathcal{F}(S,F)}{g})(\overset{\in S}{s}) \overset{\text{def}}{=} f(s) + g(s) \; ; \qquad (\underset{\in\,F}{a} f)(s) \overset{\text{def}}{=} a f(s).$$

- $\mathbb{C}$ is a vector space over $F = \mathbb{C}$

<u>Question</u>: Is $V = \mathbb{R}$ a vector space over $F = \mathbb{C}$??

- Consider the differential equation:

$$(*) \quad \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \qquad (a, b \in \mathbb{R})$$

Let S be the set of twice differentiable functions on $\mathbb{R}$ satisfying $(*)$.

Then S is a vector space under usual addition and scalar multiplication is a vector space.

<u>Proposition:</u> Let V be a vector space over F. Then:

(a) The element $\vec{0}$ in (VS3) is unique, called <u>zero vector</u>

(b) $\forall \vec{x} \in V$, the element $\vec{y}$ in (VS4) is unique, called the <u>additive inverse</u> (Denoted as $-\vec{x}$)

(c) $\vec{x} + \vec{z} = \vec{y} + \vec{z} \Rightarrow \vec{x} = \vec{y}$ (Cancellation law)

(d) $\underset{\overset{\uparrow}{F}}{0}\vec{x} = \vec{0} \quad \forall \vec{x} \in V$

(e) $(\underset{\overset{\uparrow}{F}}{-a})\vec{x} = -(a\vec{x}) = a(-\vec{x}), \quad \forall a \in F, \forall \vec{x} \in V$

(f) $\underset{\overset{\uparrow}{F}}{a}\vec{0} = \vec{0} \quad \forall a \in F$

Subspace

Definition: A subset W of a vector space V over a field F is called a <u>subspace</u> of V if W is a vector space over F under the same addition and scalar multiplication inherited from V.

Proposition: Let V be a vector space V over F. A subset $W \subset V$ is a subspace iff the following 3 conditions hold:

(a) $\vec{0}_V \in W$

(b) $\vec{x} + \vec{y} \in W$, $\forall \vec{x}, \vec{y} \in W$ (closed under $+$)

(c) $a\vec{x} \in W$, $\forall a \in F$, $\forall \vec{x} \in W$ (closed under $\cdot$)

Examples:

- For any vector space V / F,

  $\{\vec{0}\} \subset V$; $V \subset V$ (trivial subspaces)

  "zero subspace"

- For $V = M_{n \times n}(F)$,

  $W_1 = \{\text{diagonal matrices}\} \subset V$ subspace

  $W_2 = \{A \in M_{n\times n}(F) : \det(A) = 0\} \subset V$ is NOT subspace.

  $(\det(A+B) \neq \det(A) + \det(B))$

  $W_3 = \{A \in M_{n\times n}(F) : \operatorname{tr}(A) = 0\} \subset V$

  subspace

- For $V = P(F)$

$P_n(F) \overset{\text{def}}{=} \{ f \in P(F) : \deg(f) \leq n \}$ is a subspace

$W \overset{\text{def}}{=} \{ f \in P(F) : \deg(f) = n \}$ is NOT subspace.

- Consider $V = F^n = \{ (x_1, x_2, \dots, x_n) : x_j \in F \text{ for } j = 1, 2, \dots, n \}$

$\overset{\parallel}{\vec{x}^T}$

Consider linear system:

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + \dots \qquad + a_{2n} x_n = b_2 \\ \vdots \qquad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \end{cases} \iff A\vec{x} = \vec{b}$$

gives a subset, the solution set $S \subset V$

Is S a subspace?

Yes if $(b_1, b_2, \dots, b_m) = \vec{0}$ (Null space / kernel)

No if $(b_1, b_2, \dots, b_m) \neq \vec{0}$ $\left(\begin{matrix} A\vec{x} = \vec{b} \\ A\vec{y} = \vec{b} \end{matrix} \Rightarrow A(\vec{x}+\vec{y}) = 2\vec{b} \right)$

<u>Theorem:</u> Any intersection of subspaces of a vector space V is a subspace of V.

<u>Question:</u> W_1 = subspace ; W_2 = subspace

$$\Downarrow$$

$W_1 \cap W_2$ is subspace

Is $W_1 \cup W_2$ a subspace?? No general.

Linear combination and Span

Definition: Let V be a vector space over F and $S \subset V$ a non-empty subset.

- We say a vector $\vec{v} \in V$ is a <u>linear combination</u> of vectors of S if $\exists\ \vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$ and $a_1, a_2, \dots, a_n \in F$ such that:

$$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n.$$

Remark: $\vec{v}$ is usually called a linear combination of $\vec{u}_1, \dots, \vec{u}_n$ and $a_1, \dots, a_n$ are the coefficients of the linear combination.

- The <u>span</u> of S, denoted as $\mathrm{Span}(S)$, is the set of all linear combination of vectors of S.

$$\mathrm{Span}(S) \overset{\text{def}}{=} \{ a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n : a_j \in F,\ \vec{u}_j \in S \text{ for } j = 1, 2, \dots, n,\ n \in \mathbb{N} \}$$

Remark: • By convention, $\text{span}(\underset{\substack{\| \\ \text{empty set}}}{\phi}) \stackrel{\text{def}}{=} \{\vec{0}\}$.

Example:

- $F^n = \operatorname{Span}(\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\})$ where $\vec{e}_j = (0, 0, \ldots, \underset{\substack{\uparrow \\ j^{th}}}{1}, 0, \ldots 0)$

- $P(F) = \operatorname{Span}(\{1, x, x^2, \ldots, x^n, \ldots\})$

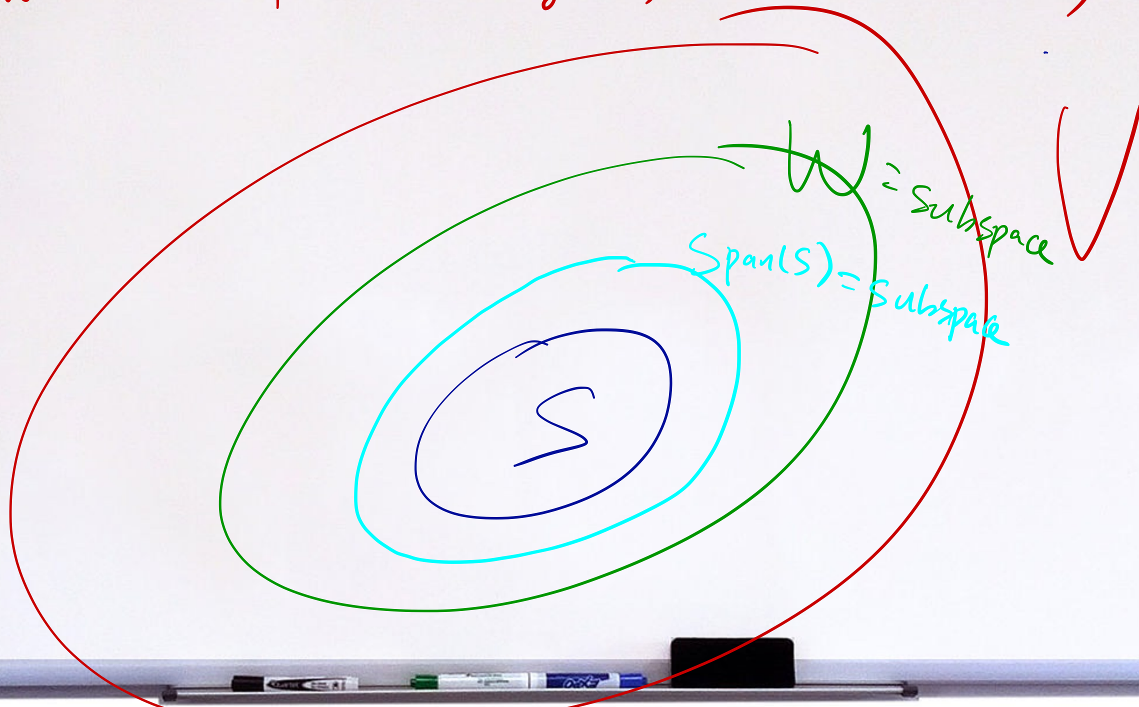
- $M_{n \times n}(F) = \operatorname{Span}(S)$

$$S = \left\{ E_{ij} \overset{def}{=} \begin{array}{cl} \begin{array}{cccc} & & \downarrow j^{th} & \end{array} & \\ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \\ 0 & \cdots & 1 & 0 \\ & & & \\ 0 & \cdots & & 0 \end{pmatrix} & \leftarrow i^{th} \end{array} : 1 \leq i, j \leq n \right\}$$

Theorem: Let $S \subset V$ be a subset of a vector space V over F.

Then, span(S) is the smallest [1] subspace [2] of V consisting S.

(If W is a subspace containing S, then span$(S) \subset W$)



Linear independence

Definition: Let V be a vector space over F. A subset $S \subset V$ is said to be **linearly dependent** if $\exists$ distinct $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$, not all zero, s.t.

$$a_1 \vec{u}_1 + a_2 \vec{u}_2 + \cdots + a_n \vec{u}_n = \vec{0}$$

Otherwise, it is said to be **linearly independent**.

e.g.

- The empty set $\phi \subset V$ is linearly independent.
- If $\vec{0} \in S$, the S is linearly dependent $\left(5 \underbrace{\vec{0}}_{\in S} = \vec{0} \right)$
- If $S = \{\vec{u}\}$ and $\vec{u} \neq \vec{0}$, then S is linearly independent.

$$\left(\begin{array}{l} \overset{\neq \vec{0}}{\lambda \vec{u}} = \vec{0} \\ \Rightarrow \lambda = 0 \end{array} \right)$$

Proposition: Let $S \subset V$ be a subset of a vector space V. Then, the following are equivalent.

(1) S is linearly independent

(2) Each $\vec{x} \in \operatorname{span}(S)$ can be expressed in a unique way as a linear combination of vectors of S.

(3) The only representations of $\vec{0}$ as linear combinations of vectors of S are trivial representations, i.e., if

$$\vec{0} = a_1\vec{u}_1 + \dots + a_n\vec{u}_n \text{ for}$$

some $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$, $a_1, a_2, \dots, a_n \in F$, then we must have $a_1 = a_2 = \dots = a_n = 0$

Example: • For $k = 0, 1, 2, \dots, n$, let $f_k(x) = 1 + x + x^2 + \dots + x^k$.

Then: $S = \{ f_0(x), f_1(x), f_2(x), \dots, f_n(x) \} \subset P_n(F)$ is a linearly independent subset.

$$0 = \vec{0} = a_0 f_0(x) + a_1 f_1(x) + \dots + a_n f_n(x)$$

$$= a_0 + a_1(1+x) + a_2(1+x+x^2) + \dots + a_n(1+x+\dots+x^n)$$

$$= (a_0 + a_1 + \dots + a_n)1 + (a_1 + a_2 + \dots + a_n)x$$

$$+ (a_2 + a_3 + \dots + a_n)x^2 + \dots + a_n x^n$$

$$\begin{cases} a_0 + a_1 + \dots + a_n = 0 \\ a_1 + \dots + a_n = 0 \\ a_2 + \dots + a_n = 0 \\ \quad\vdots \\ a_n = 0 \end{cases} \Rightarrow a_1 = a_2 = \dots = a_n = 0.$$

Theorem: Let S be a linearly independent subset of a vector space V. Let $\vec{v} \in V \setminus S$. Then: $S \cup \{\vec{v}\}$ is linearly dependent iff $\vec{v} \in \operatorname{Span}(S)$.

Definition: A basis for a vector space V is a subset $\beta \subset V$ such that:

- β is linearly independent and
- β spans V, i.e. $\mathrm{Span}(\beta) = V$.

e.g.

- F^n: $\{\vec{e}_1 = (1, 0, \dots, 0), \vec{e}_2 = (0, 1, 0 \dots, 0), \dots, \vec{e}_i = (0, \dots, 0, \overset{\text{i-th}}{1}, 0 \dots 0), \dots, \vec{e}_n = (0, 0, \dots, 1)\}$ is a basis for F^n. (Standard basis)
- $M_{2\times2}(F)$: $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \subset M_{2\times2}(F)$ is a basis for $M_{2\times2}(F)$
- $\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(F)$
- $\{1, x, x^2, \dots\}$ is a basis for $P(F)$.

Theorem: Let V be a vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\} \subset V$.

Then: β is basis for V if and only if: $\forall \vec{v} \in V$, $\exists!$ (unique)

(for all) (in) (there exist)

$a_1, a_2, \ldots, a_n \in F$ such that:

$$\vec{v} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \cdots + a_n \vec{u}_n.$$

V with $\beta = \{$  ,  ,  $\}$

$\in V$

Pineapple is associated with a unique 2, 3, 4 such that

Pineapple = 2  + 3  + 4 

$$\text{Pineapple} \longleftrightarrow \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{R}^3$$

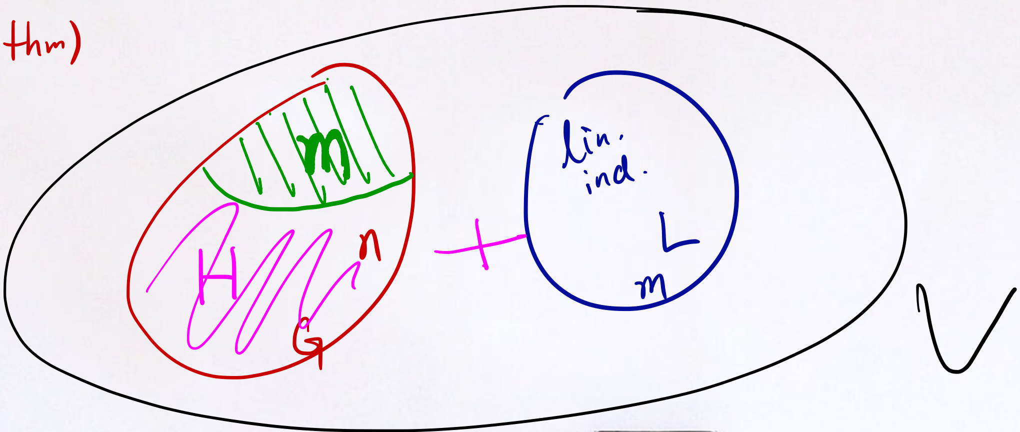
Lemma: Let S be a linearly dependent subset of a vector space V.
Then: $\exists\, \vec{v} \in S$ such that $\operatorname{span}(S \setminus \{\vec{v}\}) = \operatorname{span}(S)$.

Theorem: Suppose S is a finite spanning set for a vector space V.
Then: $\exists\, \beta \subset S$ which is a basis for V.
(A finite spanning set can be reduced to a basis)

Theorem: Let V be a vector space.
Let $G \subset V$ be a spanning set for V consisting of n vectors.
and $L \subset V$ be a linearly independent subset consisting of m vectors.

Then, $m \leq n$ and $\exists\, H \subset G$ consisting of exactly $n-m$ vectors
such that $L \cup H$ spans V.

(Replacement thm)



Dimension

Cor 1: Let V be a vector space having a finite basis. Then, every basis of V contains the same number of vectors.

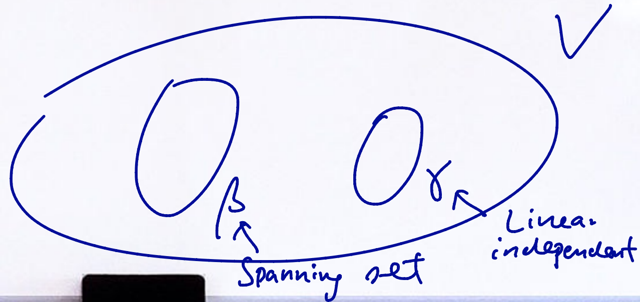
Pf: Let β and γ be two bases of V.

Since β spans V and γ is lin. independent,

then $|\gamma| \leq |\beta|$ (by replacement Thm)

Similarly, $|\beta| \leq |\gamma|$

$\Rightarrow |\gamma| = |\beta|$.



Definition: A vector space V is called <u>finite-dimensional</u> if it has a finite basis. The <u>dimension</u> of V, denoted as $\dim(V)$, is the number of vectors in a basis for V.

A vector space which is not finite-dimensional is called <u>infinite-dimensional</u>

Example:

- F^n is n-dimensional
- $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is infinite-dimensional