Lecture 11: Eigenvalue & Eigenvectors

Def: A linear operator $T : V \to V$ (where V is finite-dim) is called diagonalizable if $\exists$ an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.
A square matrix A is called diagonalizable if L_A is so.

Observation: Say $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

If $D = [T]_\beta$ is diagonal, then $\forall \vec{v}_j \in \beta$, we have:

$$D = (D_{ij}) \qquad T(\vec{v}_j) = \sum_{i=1}^{n} D_{ij}\vec{v}_i = \underbrace{D_{jj}}_{\lambda_j}\vec{v}_j = \lambda_j \vec{v}_j$$

Conversely, if $T(\vec{v}_j) = \lambda_j \vec{v}_j$ for some $\lambda_1, \lambda_2, \dots, \lambda_n \in F$,

then:

$$[T]_\beta = \begin{pmatrix} | & & \\ [T(\vec{v}_1)]_\beta & - & - \\ | & & \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & \lambda_n \end{pmatrix}$$

Def: Let T be a linear operator on a vector space V/F.
A non-zero vector $\vec{v} \in V$ is called an eigenvector of T
if $\exists\, \lambda \in F$ s.t. $T(\vec{v}) = \lambda \vec{v}$. In this case, $\lambda \in F$
is called an eigenvalue corresponding to the eigenvector $\vec{v}$.

For a square matrix $A \in M_{n\times n}(F)$, a non-zero vector $\vec{v} \in F^n$
is called an eigenvector of A if it is an eigenvector of L_A.

That is: $A\vec{v} = \lambda \vec{v}$ for some $\lambda \in F$.

λ is called the eigenvalue corresponding to the eigenvector $\vec{v}$.

Prop: A linear operator $T: V \to V$ ($V =$ fin-dim) is diagonalizable iff $\exists$ an ordered basis β for V consisting of eigenvectors of T.

In such case, if $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then:

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

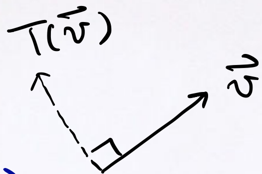
where λ_j is the eigenvalue of T corresponding to $\vec{v}_j$.

Example: • $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $\beta = \underbrace{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}}_{\text{Check that they are All eigenvectors and } \beta \text{ is basis.}}$

Then: $[L_A]_\beta = \begin{pmatrix} 4 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

• Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\frac{\pi}{2}$ in counter-clockwise direction.

(Check: $T = L_A$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)



Then: $T(\vec{v})$ is always perpendicular to $\vec{v}$.

$\therefore$ For $\forall \vec{v} \neq \vec{0}$, it cannot be an eigenvector because: $T(\vec{v}) \neq \lambda \vec{v}$ for some $\lambda \in F$

<u>Example</u>: Consider $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by:

Space of smooth function

are infinitely differentiable

$$T(f) = f'$$

Then an eigenvector of T with eigenvalue λ is a non-zero solution of:

$$\frac{df}{dt} = \lambda f(t)$$

$$\iff f(t) = C e^{\lambda t} \text{ for some constant } C.$$

$\therefore$ all $\lambda \in \mathbb{R}$ is an eigenvalue of T.

Prop: Let $A \in M_{n\times n}(F)$. Then $\lambda \in F$ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$.

Pf: $\lambda \in F$ is an eigenvalue of A

$\Longleftrightarrow$ $\exists\, \vec{v} \in F^n \setminus \{\vec{0}\}$ s.t. $A\vec{v} = \lambda \vec{v}$.

$\Longleftrightarrow$ $(A - \lambda I_n)\vec{v} = \vec{0}$

$\Longleftrightarrow$ $A - \lambda I_n$ is singular

$\Longleftrightarrow$ $A - \lambda I_n$ is not invertible

$\Longleftrightarrow$ $\det(A - \lambda I_n) = 0$.

Def: The characteristic polynomial of $A \in M_{n\times n}(F)$ is defined as the polynomial $f_A(t) \overset{\text{def}}{=} \det(A - tI_n) \in P_n(F)$

Def: Let T be a linear operator on an n-dim vector space V. Choose an ordered basis β for V. Then, the characteristic polynomial of T is defined as the characteristic polynomial of $[T]_\beta$.

(i.e. $f_T(t) \overset{\text{def}}{=} \det([T]_\beta - tI_n) \in P_n(F)$)

Prop: $f_T(t)$ is well-defined, i.e. independent of the choice of β.

Pf: If β' is another ordered basis for V, then:

$$[T]_{\beta'} = Q^{-1}[T]_{\beta} Q \quad \left(Q = [I_V]_{\beta'}^{\beta} \right)$$

Then:

$$\begin{aligned} \det([T]_{\beta'} - tI_n) &= \det(Q^{-1}[T]_{\beta} Q - tI_n) \\ &= \det(Q^{-1}([T]_{\beta} - tI_n)Q) \\ &= \underbrace{\cancel{\det(Q^{-1})}}_{= \frac{1}{\det(Q)}} \det([T]_{\beta} - tI_n) \cancel{\det(Q)} \\ &= f_T(t). \end{aligned}$$

Prop: Let $A \in M_{n\times n}(F)$. Then:

- $f_A(t)$ is of degree n and with leading coefficient $(-1)^n$.
- A has at most n distinct eigenvalues.

Pf: Exercise

Def: A polynomial $f(t) \in P(F)$ splits over F if $\exists$ $c, a_1, a_2, \dots, a_n \in F$ s.t. $f(t) = c(t-a_1)(t-a_2)\dots(t-a_n)$

Prop: The characteristic polynomial of a diagonalizable linear operator on a finite-dim vector space V/F splits over F.

Pf: If V is a n-dim and $T: V \to V$ is diagonalizable,

then $\exists$ a basis $\beta \subset V$ s.t.

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Then: $f_T(t) = \det([T]_\beta - t I_n)$

$$= (-1)^n (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)$$

Prop: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T. Then, $\vec{v} \in V$ is an eigenvector of T corresponding to λ iff:

$$\vec{v} \in N(T - \lambda I_V) \setminus \{\vec{0}\}$$

Pf: Exercise.

$$T\vec{v} = \lambda\vec{v}$$

$$\Leftrightarrow (T - \lambda I_V)\vec{v} = \vec{0}$$

Def: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T.

Then: the subspace $E_\lambda \overset{def}{:=} N(T - \lambda I_V) = \{\vec{x} \in V : T(\vec{x}) = \lambda\vec{x}\} \subset V$

is called the <u>eigenspace</u> of T corresponding to λ.

Eigenspaces of a matrix $A \in M_{n\times n}(F)$ is defined as those of L_A

Prop: Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors of T corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively, then: $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are linearly independent.

Proof: We prove by induction on k.

For $k=1$, $\vec{v}_1 \neq \vec{0} \Rightarrow \{\vec{v}_1\}$ is lin. independent.

Suppose the statement holds for $k \geq 1$ distinct eigenvalues.

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}$ be eigenvectors corresponding to $k+1$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}$ of T.

If $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k + a_{k+1}\vec{v}_{k+1} = \vec{0}$ for $a_i \in F$,

then applying $T - \lambda_{k+1} I_V$ to both sides gives:

$\vec{v}_{k+1} \in N(T - \lambda_{k+1} I_V) \setminus \{\vec{0}\}$

$$a_1(\lambda_1 - \lambda_{k+1})\vec{v}_1 + \dots + a_k(\lambda_k - \lambda_{k+1})\vec{v}_k = \vec{0}$$

By induction hypothesis,

$$a_1(\underbrace{\lambda_1 - \lambda_{k+1}}_{\neq 0}) = \dots = a_k(\underbrace{\lambda_k - \lambda_{k+1}}_{\neq 0}) = 0$$

$\Rightarrow a_1 = a_2 = \dots = a_k = 0$

$\Rightarrow a_{k+1}\underbrace{\vec{v}_{k+1}}_{\neq \vec{0}} = \vec{0}$

$\Rightarrow a_{k+1} = 0$

$\therefore \{\vec{v}_1, \dots, \vec{v}_{k+1}\}$ is lin. indep.