MATH 2028 Honours Advanced Calculus II
2023-24 Term 1
Problem Set 9

due on Dec 1, 2023 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: We will use $\mathbb{A}^k(\mathbb{R}^n)$ to denote the space of differential k-forms on $\mathbb{R}^n$.

Problems to hand in

1. Let $n = (n_1, n_2, n_3) \in \mathbb{R}^3$ be a unit vector and $v, w \in \mathbb{R}^3$ be orthogonal to n. Let
$$\omega = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy.$$
Prove that $\omega(v, w)$ is the signed area of the parallelogram spanned by v and w (the sign being determined by whether $\{n, v, w\}$ forms a right-handed orthonormal basis for $\mathbb{R}^3$).

2. We say that a k-form is *closed* if $d\omega = 0$ and *exact* if $\omega = d\eta$ for some $(k-1)$-form η.
   (a) Prove that an exact form is closed. Is every closed form exact?
   (b) Prove that if ω and ϕ are closed, then $\omega \wedge \phi$ is closed.
   (c) Prove that if ω is exact and ϕ is closed, then $\omega \wedge \phi$ is exact.

3. Compute the area of the surface in $\mathbb{R}^4$ parametrized by
$$g(u, v) = (u, v, u^2 - v^2, 2uv)$$
with $(u, v) \in \mathbb{R}^2$ satisfying $u^2 + v^2 \leq 1$.

4. (a) Suppose M and M' are two compact oriented k-dimensional submanifolds of $\mathbb{R}^n$ with boundary, and suppose $\partial M = \partial M'$. Prove that for any $(k-1)$ form ω, we have
$$\int_M d\omega = \int_{M'} d\omega.$$
   (b) Use (a) to compute $\int_M d\omega$ where M is the upper hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, oriented with outward-pointing normal having positive z-component and
$$\omega = (x^3 + 3x^2 y - y)\, dx + (y^3 z + x + x^3)\, dy + (x^2 + y^2 + z)\, dz.$$

Suggested Exercises

1. Suppose $\omega \in \Lambda^k(\mathbb{R}^n)^*$ and k is odd. Prove that $\omega \wedge \omega = 0$. Give an example to show that it does not hold when k is even.

2. Let $v, w \in \mathbb{R}^3$. Prove that $dx(v \times w) = dy \wedge dz(v, w)$, $dy(v \times w) = dz \wedge dx(v, w)$ and $dz(v \times w) = dx \wedge dy(v, w)$.

3. Can there be a function f so that df is the given 1-form ω (everywhere ω is defined)? If so, find f.

   (a) $\omega = -y\ dx + x\ dy$
   (b) $\omega = 2xy\ dx + x^2\ dy$
   (c) $\omega = y\ dx + z\ dy + x\ dz$
   (d) $\omega = (x^2 + yz)\ dx + (xz + \cos y)\ dy + (z + xy)\ dz$
   (e) $\omega = \frac{x}{x^2+y^2}\ dx + \frac{y}{x^2+y^2}\ dy$
   (f) $\omega = -\frac{y}{x^2+y^2}\ dx + \frac{x}{x^2+y^2}\ dy$

4. For each of the following k-forms ω, can there be a $(k-1)$-form η (defined wherever ω is) so that $d\eta = \omega$?

   (a) $\omega = dx \wedge dy$
   (b) $\omega = x\ dx \wedge dy$
   (c) $\omega = z\ dx \wedge dy$
   (d) $\omega = z\ dx \wedge dy + y\ dx \wedge dz + z\ dy \wedge dz$
   (e) $\omega = x\ dx \wedge dy + y\ dx \wedge dz + z\ dy \wedge dz$
   (f) $\omega = (x^2 + y^2 + z^2)^{-1}(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$

5. Define $* : \mathcal{A}^1(\mathbb{R}^3) \to \mathcal{A}^2(\mathbb{R}^3)$ by

$$*(dx) = dy \wedge dz, \quad *(dy) = dz \wedge dx \quad \text{and} \quad *(dz) = dx \wedge dy,$$

   extending by linearity. If f is a smooth function, show that

$$d * (df) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz.$$

6. Suppose $\omega \in \mathcal{A}^1(\mathbb{R}^n)$ and there is a nowhere vanishing function λ so that $\lambda\omega = df$ for some f. Prove that $\omega \wedge d\omega = 0$.

7. Let $g(\rho, \phi, \theta) : (0, \infty) \times (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ be the spherical coordinates map, i.e.

$$g(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

   Compute $g^*(dx \wedge dy \wedge dz)$.

8. In each of the following, compute the pullback $g^*\omega$ and verify that $g^*(d\omega) = d(g^*\omega)$:

   (a) $g(v) = (3\cos 2v, 3\sin 2v)$, $\omega = -y\ dx + x\ dy$
   (b) $g(u, v) = (\cos u, \sin u, v)$, $\omega = z\ dx + x\ dy + y\ dz$
   (c) $g(u, v) = (\cos u, \sin v, \sin u, \cos v)$, $\omega = (-x_3\ dx_1 + x_1\ dx_3) \wedge (-x_2\ dx_4 + x_4\ dx_2)$

9. Suppose that $k \leq n$. Let $\omega_1, \cdots, \omega_k \in (\mathbb{R}^n)^*$ and suppose that $\sum_{i=1}^k dx_i \wedge \omega_i = 0$. Prove that there exist $a_{ij} \in \mathbb{R}$ such that $a_{ji} = a_{ij}$ and $\omega_i = \sum_{j=1}^k a_{ij}\ dx_j$.

10. Suppose $U \subset \mathbb{R}^m$ is open and $g : U \to \mathbb{R}^n$ is smooth. Prove that for any $\omega \in \mathcal{A}^k(\mathbb{R}^n)$ and $v_1, \cdots, v_k \in \mathbb{R}^m$, we have

$$g^*\omega(a)(v_1, \cdots, v_k) = \omega(g(a))(Dg(a)v_1, \cdots, Dg(a)v_k).$$

11. Check that the boundary orientation on $\partial\mathbb{R}^k_+$ is $(-1)^k$ times the usual orientation on $\mathbb{R}^{k-1}$.

12. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + 3y - z = 1$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate

$$\int_C y\ dx - 2z\ dy + x\ dz$$

directly and by applying Stokes' Theorem.

13. Compute $\int_C (y - z)\ dx + (z - x)\ dy + (x - y)\ dz$ where C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $\frac{x}{a} + \frac{z}{b} = 1$, oriented clockwise as viewed from high above the xy-plane.

14. Let C be the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate

$$\int_C 2z\ dx + 3x\ dy - dz.$$

15. Let $\Omega \subset \mathbb{R}^3$ be the region bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the plane $z = 0$. Compute

$$\int_{\partial\Omega} xz\ dy \wedge dz + yz\ dz \wedge dx + (x^2 + y^2 + z^2)\ dx \wedge dy$$

directly and by applying Stokes' Theorem.

16. Let $\omega = y^2\ dy \wedge dz + x^2\ dz \wedge dx + z^2\ dx \wedge dy$, and M be the solid paraboloid $0 \le z \le 1 - x^2 - y^2$. Evaluate $\int_{\partial M} \omega$ directly and by applying Stokes' Theorem.

17. Let M be the surface of the paraboloid $z = 1 - x^2 - y^2 \ge 0$, oriented so that the outward-pointing normal has positive z-component. Let $F(x, y, z) = (x^2 z, y^2 z, x^2 + y^2)$. Compute $\int_M F \cdot \vec{n}\ d\sigma$ directly and by applying Stokes' Theorem.

18. Compute $\int_M d\omega$ where M is the portion of the paraboloid $z = x^2 + y^2$ lying beneath $z = 4$, oriented with outward-pointing normal having positive z-component, and $\omega = y\ dx + z\ dy + x\ dz$.

19. Let $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4\ :\ x_1^2 + x_2^2 + x_3^2 \le x_4 \le 1\}$, with the standard orientation inherited from $\mathbb{R}^4$. Evaluate

$$\int_{\partial M} (x_1^3 x_2^4 + x_4)\ dx_1 \wedge dx_2 \wedge dx_3.$$

20. Let S be the portion of the cylinder $x^2 + y^2 = a^2$ lying above the xy-plane and below the sphere $x^2 + (y - a)^2 + z^2 = 4a^2$. Let C be the intersection of the cylinder and sphere, oriented clockwise as viewed from high above the xy-plane.

    (a) Evaluate $\int_S z\ d\sigma$.

    (b) Use (a) to compute $\int_C y(z^2 - 1)\ dx + x(1 - z^2)\ dy + z^2\ dz$.

21. Let C be the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate $\int_C z^3 \, ds$.

Challenging Exercises

1. Prove that there is a unique linear operator $d : \mathcal{A}^k(\mathbb{R}^n) \to \mathcal{A}^{k+1}(\mathbb{R}^n)$ for all k such that

   (1) $df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \, dx_j$ for all functions $f : \mathbb{R}^n \to \mathbb{R}$

   (2) $d(f\omega) = df \wedge \omega + f \, d\omega$ for all functions $f : \mathbb{R}^n \to \mathbb{R}$ and $\omega \in \mathcal{A}^k(\mathbb{R}^n)$

   (3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for any $\omega \in \mathcal{A}^k(\mathbb{R}^n)$, $\eta \in \mathcal{A}^\ell(\mathbb{R}^n)$

   (4) $d(d\omega) = 0$ for all $\omega \in \mathcal{A}^k(\mathbb{R}^n)$

2. Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function whose graph is the surface S.

   (a) Consider the area 2-form σ on S given by

   $$\sigma = \frac{1}{\sqrt{1 + |\nabla f|^2}} \left(-\frac{\partial f}{\partial x} \, dy \wedge dz - \frac{\partial f}{\partial y} \, dz \wedge dx + dx \wedge dy \right).$$

   Show that $d\sigma = 0$ if and only if f satisfies the minimal surface equation:

   $$\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) \frac{\partial^2 f}{\partial x^2} - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} + \left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right) \frac{\partial^2 f}{\partial y^2} = 0.$$

   (b) Show that for any compact oriented surface $N \subset \mathbb{R}^3$, we have

   $$\int_N \sigma \leq \text{area}(N)$$

   and equality holds if and only if N is parallel to S.

   (c) Suppose further that $\partial N = \partial S$. Prove that $\text{area}(S) \leq \text{area}(N)$.

3. (a) Prove that a k-dimensional submanifold with boundary $M \subset \mathbb{R}^n$ is orientable if and only if there is a nowhere-zero k-form on M.

   (b) Show that M is orientable if and only if there is a volume form globally defined on M.