MATH 2028 Honours Advanced Calculus II
2023-24 Term 1
Problem Set 6

due on Nov 8, 2023 (Wednesday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Problems to hand in

1. Calculate the line integral $\int_C F \cdot d\vec{r}$ where
   (a) $F(x,y,z) = (y^2, z, -3xy)$ where C is the line segment from $(1,0,1)$ to $(2,3,-1)$.
   (b) $F(x,y) = (-y^3, x^3)$ where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ oriented counterclockwise.
2. Let C be the curve of intersection of the upper hemisphere $x^2+y^2+z^2 = 4$, $z \geq 0$ and the cylinder $x^2 + y^2 = 2x$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where $F(x,y,z) = (y,z,x)$.
3. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ is the vector field
$$F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$
   and C is an arbitrary path from $(1,1)$ to $(2,2)$ not passing through the origin.
4. Determine which of the following vector field F is conservative on $\mathbb{R}^n$. For whose that are conservative, find a potential function f for it. For those that are not conservative, find a closed curve such that $\oint_C F \cdot d\vec{r} \neq 0$.
   (a) $F(x,y) = (y^2, x^2)$;
   (b) $F(x,y,z) = (y^2 z, 2xyz + \sin z, xy^2 + y \cos z)$.
5. Find the area of the region enclosed by the curve $x^{2/3} + y^{2/3} = 1$.

Suggested Exercises

1. Calculate the line integral $\int_C F \cdot d\vec{r}$ where $F(x,y) = (y,x)$ and C is the following parametrized curve:
   (a) $\gamma(t) = (t,t)$, $0 \leq t \leq 1$;
   (b) $\gamma(t) = (t,t^2)$, $0 \leq t \leq 1$;
   (c) $\gamma(t) = (1-t, 1-t)$, $0 \leq t \leq 1$;
   (d) $\gamma(t) = (\cos^2 t, 1 - \sin^2 t)$, $0 \leq t \leq \frac{\pi}{2}$;

(e) $\gamma(t) = (\sin 2t, 1 - \cos 2t)$, $0 \le t \le \frac{\pi}{4}$;

(f) $\gamma(t) = (\cos t, 1 - \sin t)$, $0 \le t \le \frac{\pi}{2}$.

2. Repeat the exercise above with the vector field $F(x, y) = (y^2, x)$.

3. Calculate the line integral $\int_C F \cdot d\vec{r}$ where

   (a) $F(x, y, z) = (z, x, y)$ and C is the line segment from $(0, 1, 2)$ to $(1, -1, 3)$.

   (b) $F(x, y, z) = (y, 0, 0)$ where C is the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$, oriented counterclockwise as viewed from high above the xy-plane.

4. Determine which of the following vector field F is conservative on $\mathbb{R}^n$. For whose that are conservative, find a potential function f for it. For those that are not conservative, find a closed curve such that $\oint_C F \cdot d\vec{r} \ne 0$.

   (a) $F(x, y) = (x + y, x + y)$;

   (b) $F(x, y) = (e^x + 2xy, x^2 + y^2)$;

   (c) $F(x, y, z) = (x^2 + y + z, x + y^2 + z, x + y + z^2)$.

5. Calculate $\int_C F \cdot d\vec{r}$ where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$F(x, y, z) = \left(3x + y^2 + 2xz, 2xy + ze^{yz} + y, x^2 + ye^{yz} + ze^{z^2}\right)$$

   and C is the parametrized curve $\gamma : [0, 1] \to \mathbb{R}^3$ given by

$$\gamma(t) = \left(e^{t^7 \cos(2\pi t^{21})}, t^{17} + 4t^3 - 1, t^4 + (t - t^2)e^{\sin t}\right).$$

6. Compute the line integral $\int_C F \cdot d\vec{r}$ where

   (a) $F(x, y) = (xy^3, 0)$ and C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise;

   (b) $F(x, y) = (-y\sqrt{x^2 + y^2}, x\sqrt{x^2 + y^2})$ and C is the circle $x^2 + y^2 = 2x$ oriented counterclockwise.

7. Let C be the circle $x^2 + y^2 = 2x$ oriented counterclosewise. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where

$$F(x, y) = \left(-y^2 + e^{x^2}, x + \sin(y^3)\right).$$

8. Find the area of the region enclosed by the curve

$$\gamma(t) = \big(\cos t + t \sin t, \sin t - t \cos t\big), \quad 0 \le t \le 2\pi$$

   and the line segment from $(1, -2\pi)$ to $(1, 0)$.

9. Compute the line integral $\int_C F \cdot d\vec{r}$ where

   (a) $F(x, y) = (-x^2 y, xy^2)$ and C is the circle of radius $a > 0$ centered at the origin, oriented counterclosewise;

   (b) $F(x, y) = (-y^2, x^2)$ and C is the boundary of the region given in polar coordinates by $r \le a$, $0 \le \theta \le \pi/4$ oriented counterclosewise.

10. Let $0 < b < a$. Find the area under the curve $f(t) = (at - b\sin t, a - b\cos t)$, $0 \le t \le 2\pi$, above the x-axis.

11. Suppose C is a piecewise C^1 closed curve in $\mathbb{R}^2$ that intersects with itself finitely many times and does not pass through the origin. Show that the line integral

$$\frac{1}{2\pi}\int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

is always an integer. This is called the *winding number* of C around the origin.

Challenging Exercises

1. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field on $\mathbb{R}^n$ defined by

$$F(x_1, x_2, \cdots, x_n) = (f(r)x_1, f(r)x_2, \cdots, f(r)x_n)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function and $r := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$.

(a) Suppose f is differentiable everywhere. Prove that for all $i, j = 1, \cdots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

on $\mathbb{R}^n \setminus \{\vec{0}\}$ where F_k is the k-th component function of the vector field F.

(b) Suppose f is continuous everywhere. Prove that F is a conservative vector field on $\mathbb{R}^n$.

2. Give a direct proof of Green's theorem for

(a) a triangle with vertices $(0,0)$, $(a,0)$ and $(0,b)$,

(b) the region $\{(x,y) : a \le x \le b, g(x) \le y \le h(x)\}$ for some C^1 function $g, h : [a,b] \to \mathbb{R}$.