MATH 2028 Honours Advanced Calculus II
2023-24 Term 1
Problem Set 2

due on Sep 27, 2023 (Wednesday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through CUHK Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: Throughout this problem set, we use R to denote a rectangle in $\mathbb{R}^n$, and $B_\delta(p) \subset \mathbb{R}^n$ to denote the open ball of radius δ centered at p.

Problems to hand in

1. (a) Let $A \subset \mathbb{R}^n$ be a content zero subset. Prove that A must be bounded. Moreover, show that ∂A has measure zero and $\text{Vol}(A) = 0$.

   (b) Let $B \subset \mathbb{R}^n$ be a bounded subset of measure zero. Suppose ∂B has measure zero. Prove that $\text{Vol}(B) = 0$.

2. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be the function

$$f(x,y) = \begin{cases} 1/q & \text{if } x, y \in \mathbb{Q} \text{ and } y = p/q \text{ where } p, q \in \mathbb{N} \text{ are coprime,} \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that f is integrable and $\int_R f \, dV = 0$.

3. Let $f : R \to \mathbb{R}$ be a bounded integrable function. Suppose p is an interior point of R at which f is continuous. Prove that

$$\lim_{\delta \to 0^+} \frac{1}{\text{Vol}(B_\delta(p))} \int_{B_\delta(p)} f \, dV = f(p).$$

Suggested Exercises

1. Let $f : R \to \mathbb{R}$ be a bounded integrable function. Prove that $|f|$ is also integrable on R and $\left| \int_R f \, dV \right| \leq \int_R |f| \, dV$.

2. Let $f : \Omega \to \mathbb{R}$ be a bounded continuous function defined on a bounded subset $\Omega \subset \mathbb{R}^n$ whose boundary $\partial \Omega$ has measure zero. Suppose Ω is path-connected, i.e. for any $p, q \in \Omega$, there exists a continuous path $\gamma(t) : [0,1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Prove that there exists some $x_0 \in \Omega$ such that

$$\int_\Omega f \, dV = f(x_0) \text{Vol}(\Omega).$$

3. (a) Prove that any content zero subset $A \subset \mathbb{R}^n$ must also have measure zero.

   (b) Give an example of a measure zero subset $A \subset \mathbb{R}^2$ which does not have content zero.

   (c) Prove that if $A \subset \mathbb{R}^n$ is compact [1] and has measure zero, then A has content zero.

[1] A subset A is compact if any open cover of A has a finite subcover. The Heine-Borel Theorem says that a subset in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

(d) Suppose $\{A_i\}_{i=1}^{\infty}$ is a sequence of measure zero subsets in $\mathbb{R}^n$. Show that $\cup_{i=1}^{\infty} A_i$ also has measure zero.

4. (a) Show that the subset $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ has measure zero.
   (b) Show that $\mathbb{Q}^c \cap [0,1]$ does not have measure zero in $\mathbb{R}$.

5. Let $f : R \to \mathbb{R}$ be a bounded function. Suppose $f = 0$ except on a *closed* set B of measure zero. Prove that f is integrable and $\int_R f \, dV = 0$.

Challenging Exercises

1. The following exercise establishes the theorem that a bounded function $f : R \to \mathbb{R}$ is integrable if and only if f is continuous on R except on a set of measure zero. Let $f : R \to \mathbb{R}$ be a bounded function. For each $p \in R$ and $\delta > 0$, we define the *oscillation of f at p* as

$$o(f,p) = \lim_{\delta \to 0^+} \left(\sup_{x \in B_\delta(p) \cap R} f(x) - \inf_{x \in B_\delta(p) \cap R} f(x) \right).$$

   (a) Show that $o(f,p)$ is well-defined and non-negative. Prove that f is continuous at p if and only if $o(f,p) = 0$.
   (b) For any $\epsilon > 0$, let $D_\epsilon := \{p \in R \;:\; o(f,p) \geq \epsilon\}$. Show that D_ϵ is a closed subset and the set of discontinuities D of f is given as $D = \cup_{n=1}^{\infty} D_{1/n}$.
   (c) Suppose f is integrable on R. Prove that $D_{1/n}$ has content zero for any $n \in \mathbb{N}$. Hence, show that D has measure zero.
   (d) Suppose D has measure zero, prove that f is integrable on R.

2. This exercise requires some familiarity with linear algebra at the level of MATH 2040/2048.
   (a) Let $A \subset \mathbb{R}^n$ be a subset of content zero. Show that for any $\epsilon > 0$, there exists finitely many cubes [2] $C_1, \cdots, C_n$ such that $A \subset \cup_{i=1}^{n} C_i$ and $\sum_{i=1}^{n} \mathrm{Vol}(C_i) < \epsilon$.
   (b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that $T(A)$ has content zero if $A \subset \mathbb{R}^n$ has content zero.
   (c) Let $F : U \to \mathbb{R}^n$ be a C^1 map from an open subset $U \subset \mathbb{R}^m$ where $m < n$. Prove that $F(A)$ has content zero (in $\mathbb{R}^n$) if $A \subset U$ is a compact subset.

[2] A cube is a rectangle with sides of equal length.