MATH 2028 Honours Advanced Calculus II
2023-24 Term 1
Problem Set 1

due on Sep 20, 2023 (Wednesday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through CUHK Blackboard on/before the due date. Please remember to write down your name and student ID. You can refer to the webpage under "Useful Links" below about how to submit assignments through Blackboard. **No late homework will be accepted.**

Notations: We use R to denote a rectangle in $\mathbb{R}^n$ throughout this problem set.

Problems to hand in

1. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be a bounded function defined by

$$f(x,y) := \begin{cases} 1 & \text{if } y < x, \\ 0 & \text{if } y \geq x. \end{cases}$$

   Prove, using the definition, that f is integrable and find $\int_R f \, dV$.

2. Let $f, g : R \to \mathbb{R}$ be bounded integrable functions. Prove that $f + g$ is integrable on R and

$$\int_R (f+g) \, dV = \int_R f \, dV + \int_R g \, dV.$$

3. (a) Suppose $f : R \to \mathbb{R}$ is a non-negative *continuous* function such that $f(p) > 0$ at some $p \in R$. Prove that $\int_R f \, dV > 0$.

   (b) Give an example to show that (a) is false if f is not assumed to be continuous.

Suggested Exercises

1. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be a bounded function defined by

$$f(x,y) := \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

   Prove, using the definition, that f is integrable and $\int_R f \, dV = \frac{1}{2}$.

2. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be a bounded function defined by

$$f(x,y) := \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove, using the definition, that f is integrable and find $\int_R f \, dV$.

3. Let $f : R \to \mathbb{R}$ be a bounded integrable function defined on a rectangle $R \subset \mathbb{R}^n$. Suppose $g : R \to \mathbb{R}$ is a bounded function such that $g(x) = f(x)$ except for finitely many $x \in R$. Show that g is integrable and $\int_R g\, dV = \int_R f\, dV$.

Challenging Exercises

1. Let f be a bounded integrable function on R. Prove that for any $\epsilon > 0$, there exists some $\delta > 0$ such that whenever $\mathcal{P}$ is a partition of R with $\text{diam}(Q) < \delta$ for all $Q \in \mathcal{P}$, we have $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.

   Hint: Let $R = [a_1, b_1] \times \cdots [a_n, b_n]$ and $w = \max_i |b_i - a_i|$. For any partition $\mathcal{P}$ with diam$(Q) < \delta$ for all $Q \in \mathcal{P}$, if we take a refinement $\mathcal{P}'$ of $\mathcal{P}$ by adding one more grid point to $[a_i, b_i]$ for some i, then we have $L(f, \mathcal{P}') \leq L(f, \mathcal{P}) + 2M\delta w^{n-1}$ where $M > 0$ is a global bound for $|f|$.

2. Let f be a bounded integrable function on R. Prove that for any $\epsilon > 0$, there exists some $\delta > 0$ such that whenever $\mathcal{P}$ is a partition of R with $\text{diam}(Q) < \delta$ for all $Q \in \mathcal{P}$, and $x_Q \in Q$ is any arbitrarily chosen point inside $Q \in \mathcal{P}$, we have

$$\left| \sum_{Q \in \mathcal{P}} f(x_Q)\text{Vol}(Q) - \int_R f\, dV \right| < \epsilon.$$

   (The sum in the above expression is what we usually call the "Riemann sum"!)